CONSTRAINED NEVANLINNA-PICK INTERPOLATION

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By

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Abstract

We are interested in complex interpolation problems that have their origins in the work of Nevanlinna and Pick. In the 90 years since their results first appeared, Nevanlinna-Pick problems have been valuable in the development of areas of pure mathematics such as operator theory, operator algebras, harmonic analysis and complex function theory. The study of interpolation problems has also been closely tied to the development of systems theory and H-infinity control theory.

We describe how these interpolation results extend to a class of subalgebras of the algebra of bounded analytic functions on the open unit disk. The problems we will look at include as special cases interpolation theory on multiply connected domains and interpolation on embedded disks.

Our methods use duality techniques, factorization results, averaging techniques and matrix theory.

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Chapter 1

Nevanlinna-Pick Interpolation

1.1 The Nevanlinna-Pick problem

The term *Nevanlinna-Pick interpolation* describes a class of complex interpolation problems. The problem was originally studied by Pick in 1916 [35] and independently by Nevanlinna in 1919 [31]. Since there are a number of Nevanlinna-Pick interpolation problems in the literature, and since we will be concerned with more than one such type of problem, we will refer to the original theorem as the classical Nevanlinna-Pick theorem.

The statement of the classical problem (the one studied by Nevanlinna and Pick) is as follows. Given n points $z_1, \ldots, z_n$ in the open unit disk $\mathbb{D}$, and n points $w_1, \ldots, w_n$ in the disk $\mathbb{D}$ characterize, in terms of the data $z_1, \ldots, z_n, w_1, \ldots, w_n$, the existence of a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ such that $f(z_j) = w_j$.

Pick's characterization was that such a function exists if and only if the matrix

$$\left[\frac{1-w_i\overline{w_j}}{1-z_i\overline{z_j}}\right]_{i,j=1}^{n} \tag{1.1}$$

is positive (semi-definite).

Pick's original proof relied on the Schwarz lemma and an inductive argument to obtain the result. Pick also established the fact that the solution to interpolating function is unique if and only if the Pick matrix (1.1) is singular. Nevanlinna worked independently on the same problem. He used Schur's algorithm to characterize the existence of a solution to the problem and also parametrized all solutions in the case where the Pick matrix is invertible.

In order to apply operator theoretic techniques to this problem, a reformulation is required. The set of bounded analytic functions on the disk will be denoted H^∞. The norm on H^∞ is the usual supremum norm

$$\|f\|_\infty := \sup\{|f(z)| \,:\, z \in \mathbb{D}\}. \tag{1.2}$$

When endowed with this norm, H^∞ is a Banach algebra and the maximum modulus principle [7, page 134, Theorem 12] shows that a function f maps the open unit disk $\mathbb{D}$ to itself if and only if f is in the closed unit ball of H^∞. Therefore, the Nevanlinna-Pick theorem characterizes the existence of an element f in H^∞ such that $\|f\|_\infty \leq 1$ and $f(z_j) = w_j$.

In Chapter 1.2 and Chapter 1.3 we will describe in more detail the way in which operator theory and the Nevanlinna-Pick problem are related. To begin with we will describe our motivation and make some general statements about the problems we plan to look at.

We would like to generalize the classical Nevanlinna-Pick result to subalgebras of H^∞. It is too much to hope that such a generalization is possible for all subalgebras, and as we will see later on, even the simplest examples can be more complicated than one might expect.

Consider the interpolation problem for an algebra $\mathcal{A} \subseteq H^\infty$. The interpolation problem is really a question about the norm on an n-dimensional quotient of $\mathcal{A}$. We will assume that $\mathcal{A}$ is weak* closed and unital. Let $z_1, \ldots, z_n$ be n points in the unit disk $\mathbb{D}$ and let $w_1, \ldots, w_n$ be n complex scalars.

We will say that $g \in \mathcal{A}$ *interpolates* or is an *interpolating function* if $g(z_j) = w_j$. A *solution* to the interpolation problem is a function $f \in \mathcal{A}$ such that $\|f\|_\infty \leq 1$, $f(z_j) = w_j$. Therefore, a solution is an interpolating function in the closed unit ball of $\mathcal{A}$. In general, there is no reason for an interpolating function to exist.

Let $\mathcal{I}$ denote the ideal of functions that vanish at the n points $z_1, \ldots, z_n$. The norm on the quotient $\mathcal{A}/\mathcal{I}$ is given by

$$\|f + \mathcal{I}\| := \inf\{\|f + h\|_\infty \, : \, h \in \mathcal{I}\} \tag{1.3}$$

Suppose that an interpolating function $f \in \mathcal{A}$ does exist. If $g \in \mathcal{A}$ and $g(z_j) = w_j$, then $f - g$ vanishes at the n points $z_1, \ldots, z_n$ and so $f - g \in \mathcal{I}$. In fact every function in the coset $f + \mathcal{I}$ interpolates z_j to w_j for $j = 1, \ldots, n$. If the interpolation problem has a solution, say $g = f + h$ with $h \in \mathcal{I}$, then $\|f + \mathcal{I}\| \leq \|f + h\|_\infty \leq 1$. On the other hand if $\|f + \mathcal{I}\| \leq 1$ for some function f, $f(z_j) = w_j$, then Lemma 3.2.1 shows that a solution to the interpolation problem does exist. Hence, determining the existence of an interpolating function involves computing the norm of $f+\mathcal{I}$ in the quotient algebra $\mathcal{A}/\mathcal{I}$.

Let $\mathcal{H}$ be a *reproducing kernel Hilbert space* on X with kernel function K. The kernel function for $\mathcal{H}$ at the point $x \in X$ is denoted k_x. The kernel function k_x is the unique element in $\mathcal{H}$ such that

$$f(x) = \langle f, k_x \rangle , \tag{1.4}$$

for all $f \in \mathcal{H}$. The *multiplier algebra* $\text{mult}(\mathcal{H})$ is defined as the set of functions $f : X \to \mathbb{C}$ such that $fh \in \mathcal{H}$ for all $h \in \mathcal{H}$. An application of the closed graph theorem shows that the map $M_f : \mathcal{H} \to \mathcal{H}$ defined by $M_f(h) = fh$ is a bounded operator on $\mathcal{H}$. The multiplier norm of f is defined by $\|f\|_{\text{mult}} := \|M_f\|$. This representation of $\text{mult}(\mathcal{H})$ on $B(\mathcal{H})$ induces a natural operator algebra norm on the multiplier algebra, i.e., there is a natural family of matrix norms on the $p \times p$ matrices over $\mathcal{A} := \text{mult}(\mathcal{H})$ given by

$$\|[f_{i,j}]\|_{M_p(\text{mult}(\mathcal{H}))} := \left\| \left[M_{f_{i,j}} \right]_{M_p} \right\|_{B(\mathcal{H}^{(p)})} . \tag{1.5}$$

The matrix norm structure of an operator algebra carries has tremendous consequences in operator algebras and in the Nevanlinna-Pick theorem. All the examples we deal with will be subalgebras of H^∞ that act as multipliers on subspaces of H^2. In this case the multiplier norm, and the corresponding matrix norms, are just the usual supremum norm, i.e.,

$$\|[f_{i,j}]\|_{M_p(\text{mult}(\mathcal{H}))} = \sup\{\|[f_{i,j}(z)]\| \, : \, z \in \mathbb{D}\}. \tag{1.6}$$

Given a point $x \in X$, $M_f^* k_x = \overline{f(x)} k_x$. This shows that f is a bounded function with $\|f\|_\infty \leq \|f\|_{\text{mult}}$, and that k_x is an eigenvector for M_f^*.

Given n points $x_1, \ldots, x_n \in X$ let $\mathcal{K}$ be the subspace spanned by $k_{x_1}, \ldots, k_{x_n}$. The orthogonal complement of $\mathcal{K}$ in $\mathcal{H}$ is the set of functions in $\mathcal{H}$ that vanish at the points $x_1, \ldots, x_n$ and we define $\mathcal{N} := \mathcal{H} \ominus \mathcal{K}$. Although $\mathcal{K}$ may not be n-dimensional, we can always reorder the kernel functions so that the set $\{k_{x_1}, \ldots, k_{x_m}\}$ is a basis for $\mathcal{K}$, with $m \leq n$.

Let $\pi_{\mathcal{K}} : \mathcal{A} \to B(\mathcal{K})$ be given by $\pi_{\mathcal{K}}(f) = P_{\mathcal{K}} M_f P_{\mathcal{K}}$, where $P_{\mathcal{K}}$ is the orthogonal projection onto $\mathcal{K}$. Since the subspace $\mathcal{K}$ is invariant for $\mathcal{A}^*$ the map $\pi_{\mathcal{K}}$ is a homomorphism. The kernel of this homomorphism is the set of functions $f \in \mathcal{A}$ such that $M_f^* k_{x_j} = \overline{f(x_j)} k_{x_j} = 0$ for $j = 1, \ldots, n$. If the kernel functions k_{x_j} are non-zero, then $f(x_j) = 0$ and we see that the kernel of this homomorphism is $\mathcal{I}$. Therefore, we have a contractive, unital representation of $\mathcal{A}/\mathcal{I}$ on $B(\mathcal{K})$. We denote this representation by $\pi_{\mathcal{K}}$ as well.

Note that $\|P_{\mathcal{K}} M_f P_{\mathcal{K}}\| \leq C$ if and only if $C^2 I - (P_{\mathcal{K}} M_f P_{\mathcal{K}})(P_{\mathcal{K}} M_f P_{\mathcal{K}})^* \geq 0$, equivalently $C^2 P_{\mathcal{K}} - P_{\mathcal{K}} M_f M_f^* P_{\mathcal{K}} \geq 0$. An element $k \in \mathcal{K}$ has the form $k = \sum_{j=1}^n \alpha_j k_{x_j}$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Using the fact that $M_f^* k_{x_j} = \overline{f(x_j)} k_{x_j}$, we see that

$$C^2 \|k\|^2 - \langle M_f M_f^* k, k \rangle = \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} (C^2 - f(x_i)\overline{f(x_j)}) K(x_i, x_j) \tag{1.7}$$

$$= \left\langle \left[(C^2 - f(x_i)\overline{f(x_j)}) K(x_i, x_j) \right] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \right\rangle. \tag{1.8}$$

Hence, $\|\pi_{\mathcal{K}}(f)\| \leq 1$ if and only if the matrix $\left[(1 - f(x_i)\overline{f(x_j)})K(x_i, x_j)\right]_{i,j=1}^{n} \geq 0$. If $\pi_{\mathcal{K}}$ is an isometry, then this last condition is equivalent to $\|f + \mathcal{I}\| \leq 1$ and the interpolation problem is solved.

As an example, consider the Hardy space H^2 of analytic functions on the disk with square summable power series. This is a reproducing kernel Hilbert space. The kernel function is the Szegö kernel K given by

$$K(z, w) = \frac{1}{1 - z\overline{w}}. \tag{1.9}$$

The multiplier algebra of H^2 is H^∞ and $\|f\|_\infty = \|f\|_{\text{mult}}$. In this case the representation $\pi_{\mathcal{K}}$ is an isometry (in fact, a complete isometry) and so the existence of a solution to the Nevanlinna-Pick problem is equivalent to the positivity of the matrix

$$\left[(1 - f(z_i)\overline{f(z_j)})K(z_i, z_j)\right]_{i,j=1}^{n} = \left[\frac{1 - w_i\overline{w_j}}{1 - z_i\overline{z_j}}\right]_{i,j=1}^{n}. \tag{1.10}$$

As it stands this reasoning is incomplete. We only know that the representation on $\mathcal{K}$ is an isometry because of the Nevanlinna-Pick theorem. However, an independent proof of the isometry would yield the Nevanlinna-Pick theorem. Sarason [37] provided such a proof.

1.2 Sarason's generalized interpolation

The connection between the Nevanlinna-Pick problem and operator theory appeared first in Sarason's seminal paper [37]. It was Sarason who reformulated the problem as a question about the distance of $f \in H^\infty$ from a weak*-closed ideal $\mathcal{I}$.

Duality techniques then give a new proof of the Nevanlinna-Pick theorem. Sarason actually proved a special case of the commutant lifting theorem which in turn implies the interpolation result.

A function $\phi \in L^2$ is called *unimodular* if and only if $|\phi| = 1$ a.e. on the circle. Equivalently, the set $\{\phi z^j : j \in \mathbb{Z}\}$ is an orthonormal basis of L^2. A unimodular function $\phi \in H^2$ is called *inner*. A function ϕ is inner if and only if the set $\{\phi z^j : j \geq 0\}$ is an orthonormal basis of H^2.

We have noted that H^∞ is the multiplier algebra of H^2. The weak*-closed (WOT-closed) subalgebra of $B(H^2)$ generated by the shift S is H^∞. Furthermore, H^∞ is equal to its own commutant and $\{S\}' = H^\infty$.

If $\mathcal{A}$ is an operator algebra on $\mathcal{H}$, then a subspace $\mathcal{M}$ of the Hilbert space $\mathcal{H}$ is called *invariant* if $A(\mathcal{M}) \subseteq \mathcal{M}$ for all $A \in \mathcal{A}$. A subspace $\mathcal{M}$ is called *semi-invariant* for $\mathcal{A}$ if there exists $\mathcal{M}_1, \mathcal{M}_2$ such that $\mathcal{M}_1, \mathcal{M}_2$ are invariant for $\mathcal{A}$ and $\mathcal{M} = \mathcal{M}_1 \ominus \mathcal{M}_2$. The key fact, as pointed out in [37], is that $\mathcal{M}$ is semi-invariant for $\mathcal{A}$ if and only if the compression of $\mathcal{A}$ to $\mathcal{M}$ is a homomorphism of $\mathcal{A}$. The subspace $\mathcal{K}$ of H^2 spanned by $k_{z_1}, \ldots, k_{z_n}$ is an example of a semi-invariant subspace of H^2 for $\mathcal{A} = H^\infty$.

When studying representations of operator algebras on Hilbert space, it is of value to classify the invariant subspaces for the algebra. Beurling's theorem [14] classifies the invariant subspaces of the shift, or equivalently the invariant subspaces of H^∞. The Helson-Lowdenslager-Srinivasan theorem [23, 39] is an extension of Beurling's theorem and classifies the invariant subspaces for the shift operator on L^p.

Theorem 1.2.1 (Beurling [14])**.** *Let $\mathcal{M}$ be a closed, non-trivial, subspace of H^2.*

The subspace $\mathcal{M}$ *is invariant for* S *if and only if there exists an inner function* ϕ *such that* $\mathcal{M} = \phi H^2$*.*

Theorem 1.2.2 (Helson-Lowdenslager-Srinivasan [21])**.** *Let* $\mathcal{M}$ *be a closed (weak***-closed if* $p = \infty$*), non-trivial, subspace of* L^p*. The subspace* $\mathcal{M}$ *is invariant for* S *if and only if one of the following two conditions is true*

1. *There exists a unimodular function* ϕ *such that* $\mathcal{M} = \phi H^p$*.*

2. *There exists a Lebesgue measurable set* $E \subseteq \mathbb{T}$*, of positive measure, such that* $\mathcal{M} = \chi_E L^p$*.*

A subspace of the form ϕH^p is called a *Beurling subspace*, a subspace of the form $\chi_E L^p$ is called a *Wiener subspace.*

If $\mathcal{I}$ is a weak*-closed ideal of H^∞, then $\mathcal{I}$ is an invariant subspace and the Helson-Lowdenslager-Srinivasan theorem tells us that $\mathcal{I} = \phi H^\infty$ where ϕ is an inner function. The main result of [37] is the following.

Theorem 1.2.3 (Sarason [37, Theorem 1])**.** *Let* ϕ *be an inner function and let* $\mathcal{K} = H^2 \ominus \phi H^2$*. If* $T \in B(\mathcal{K})$ *is an operator in the commutant of* $\pi_{\mathcal{K}}(S)$*, then there exists an operator* $R \in B(H^2)$ *such that* $RS = SR$*,* $\|R\| = \|T\|$ *and* $R|_{\mathcal{K}} = T$*.*

This tells us that compressing M_f to the semi-invariant subspace $H^2 \ominus \phi H^2$ is an isometry of $H^\infty/\mathcal{I}$. The case where ϕ is the Blaschke product with simple zeroes at $z_1, \ldots, z_n$ gives us the Nevanlinna-Pick theorem.

1.3 Abrahamse's Theorem

The Hardy space H^2 is not the only space of analytic functions on the disk that has H^∞ as its multiplier algebra. The Bergman space L_a^2 is an example of a space of analytic functions on the disk with $\mathrm{mult}(L_a^2) = H^\infty$. In some sense H^2 is the "right" space of analytic functions to consider for the Nevanlinna-Pick problem.

There is a theory of Hardy spaces on multiply connected domains in the complex plane. A thorough introduction to this subject can be found in Fisher's book [19]. There are many natural ways to define the Hardy space of a multiply connected domain R. Our eventual goal is to lift function theory from the region R to the disk $\mathbb{D}$ through the use of the universal covering map $p : \mathbb{D} \to R$. With this in mind we choose a definition that is suited to our work.

The *Hardy Hilbert space of a domain* R is defined as the set of functions f that are analytic in R with the property that $|f|^2$ has a harmonic majorant, i.e., a harmonic function u such that $|f|^2 \leq u$. We will denote this Hilbert space $H^2(R)$. If we fix a point $z \in R$, then the norm of $f \in H^2(R)$ is defined as the infimum $u(z)^{\frac{1}{2}}$ over all harmonic majorants u of $|f|^2$. This definition depends on the point z. However, as z varies over R, the induced norms are equivalent to one another. If $p : \mathbb{D} \to R$ is the universal covering map, then harmonic measure μ for the point $p(0)$ lifts to Lebesgue measure on the circle. From this we can derive the existence of boundary values for functions in the Hardy space $H^2(R)$. This reasoning can be carried through to define the Hardy spaces $H^p(R)$.

For a multiply connected domain R the Banach algebra $H^\infty(R)$ of bounded analytic functions on R is the multiplier algebra of the reproducing kernel Hilbert space $H^2(R)$. However, the representation π that one obtains by compressing a multiplier M_f to the span of the kernel functions for the points $z_1, \dots, z_n \in R$ is not isometric [27]. In this respect, Nevanlinna-Pick theory on multiply connected domains is already different from the classical result.

M. B. Abrahamse showed [1] that the Nevanlinna-Pick theorem could be generalized to multiply connected domains. His proof, like Sarason's, uses duality techniques. In order to get the correct result Abrahamse realized that one needs to consider a family of reproducing kernel Hilbert spaces $H^2_\lambda(R)$, with kernel function K^λ, indexed by $\lambda \in \mathbb{T}^m$, where m is the connectivity of the region R. The case $\lambda = (1, \dots, 1) \in \mathbb{T}^m$ corresponds to the space $H^2(R)$. These spaces are also modules over $H^\infty(R)$ and there is a representation of $\pi_\lambda : H^\infty(R)/\mathcal{I} \to B(\mathcal{K}_\lambda)$ where $\mathcal{K}_\lambda$ is the span of the kernel functions for $H^2_\lambda(R)$ at the points $z_1, \dots, z_n$. Abrahamse showed that the direct sum of this family of representations was, in fact, an isometry.

Theorem 1.3.1 (Abrahamse [1, Theorem 1]). *Let $z_1, \dots, z_n \in R$ and $w_1, \dots, w_n \in \mathbb{C}$. There exists a function $f \in H^\infty(R)$ such that $\|f\|_\infty \le 1$ and $f(z_j) = w_j$ if and only if the matrices*

$$A_\lambda := \left[(1 - w_i\overline{w_j})K^\lambda(z_i, z_j)\right]_{i,j=1}^n \ge 0 \tag{1.11}$$

for all $\lambda \in \mathbb{T}^m$.

It is not immediately clear why the theorem for multiply connected regions should require a family of kernels, while the theorem for the disk requires only one. The reason is hidden in the duality argument used by Sarason and Abrahamse.

The argument relies heavily on the ability to factor an H^1 function as the product of two H^2 functions. This is called Riesz factorization [22, Theorem 20]

Theorem 1.3.2 (Riesz factorization)**.** *Let $f \in H^1$. There exists functions $g, h \in H^2$ such that $|f|^{1/2} = |g| = |h|$ with $f = gh$.*

For multiply connected domains a similar result is true. Given a function $f \in H^1(R)$, there exists a $\lambda \in \mathbb{T}^m$, $g \in H^2_\lambda$, $h \in H^2_{\overline{\lambda}}$ such that $f = gh$, $|f|^{1/2} = |g| = |h|$. Hence, we cannot factor f and remain in the space $H^2(R)$.

For an excellent introduction to the subject of Nevanlinna-Pick interpolation consult the book by Agler and McCarthy [4].

1.4 Constrained Nevanlinna-Pick problems

We will study a class of Nevanlinna-Pick interpolation problems that have arisen in the last 10 years. In keeping with the name given to one such problem [15] we call them *constrained Nevanlinna-Pick interpolation problems*. We cite four reasons why such problems may be interesting to look at. First, the problems are simple variations on the classical Nevanlinna-Pick problem and are easily formulated. Second, their theory is varied enough from the classical case to be of interest. Third, the Nevanlinna-Pick problem for a multiply connected domain can be viewed as a special case of the constrained problems we will look at. Finally, the recent work of Agler and McCarthy [5, 6] on cusp algebras and embedded disks suggests that there are close connections between these constrained problems and interpolation problems on one-dimensional varieties in $\mathbb{C}^m$.

Suppose, just as in the classical interpolation problem, that we are given n points $z_1, \ldots, z_n \in \mathbb{D}$ and $w_1, \ldots, w_n \in \mathbb{C}$. We require that the interpolating function belong to a fixed subalgebra $\mathcal{A}$ of H^∞. It would be too much to hope that we could study interpolation problems for any subalgebra of H^∞. Therefore, we single out an interesting class of algebras that are easy to describe. Our motivating example is to be found in the work of Solazzo [38, Example 3.3.5].

Select m distinct points $a_1, \ldots, a_m \in \mathbb{D}$ and define

$$H^\infty_{a_1,\ldots,a_m} := \{f \in H^\infty \,:\, f(a_1) = \ldots = f(a_m)\}. \tag{1.12}$$

This is easily seen to be a weak*-closed, unital subalgebra of H^∞. The algebra is in fact of finite codimension in H^∞. The set of functions that vanish at the m points $a_1, \ldots, a_m$ is a weak*-closed ideal of H^∞ and it is easy to see that $H^\infty_{a_1,\ldots,a_m}$ is merely the unitization of this ideal. The factorization theory for Hardy spaces shows us that the ideal $\mathcal{I}_{a_1,\ldots,a_m}$ of functions in H^∞ that vanish at the m points $a_1, \ldots, a_m$ is equal to BH^∞, where B is the Blaschke product for the points $a_1, \ldots, a_m$, and so

$$H^\infty_{a_1,\ldots,a_m} = \mathbb{C} + BH^\infty. \tag{1.13}$$

The next example in the literature is the one studied in [15]. The algebra under consideration is the set of functions $f \in H^\infty$ such that $f'(0) = 0$. This algebra is denoted H^∞_1 and we see that

$$H^\infty_1 = \mathbb{C} + z^2 H^\infty. \tag{1.14}$$

This suggests that the study of algebras of the form $H_B^\infty := \mathbb{C} + BH^\infty$ where B is a Blaschke product could be of value. In Chapter 2.2 we will also see that there is a close connection between algebras of the form H_B^∞ and $H^\infty(R)$, where R is a multiply connected domain.

1.5 Outline of results

The work that follows deals with the Nevanlinna-Pick problem for a class of subalgebras of H^∞. The Nevanlinna-Pick problem lies at a crossroads of function theory, operator theory and operator algebras. In the past, variations on the classical Nevanlinna-Pick problem have provided insight into operator theory.

In Chapter 1.4 we will describe two subalgebras that are contained in H^∞. The first of these, H_B^∞, arises as the unitization of a weak*-closed ideal of H^∞. The second is the fixed point algebra H_Γ^∞ for the action on the disk of a Fuchsian group Γ. The latter example is directly related to the algebras $H^\infty(R)$ where R is a multiply connected domain.

We have seen that invariant subspaces and distance formulae play a central role in Nevanlinna-Pick theory and so Chapter 3.1 is dedicated to classifying the invariant subspaces for H_B^∞. Chapter 3.2 is central to our work. We prove two distance formulae that will allow us to derive the Nevanlinna-Pick theorem for the algebras H_B^∞ and H_Γ^∞.

In Chapter 4.1 we begin by proving the interpolation results for H_B^∞. Combining the distance formula for L_Γ^∞ with the a result of Forelli leads to a generalization of Abrahamse's theorem for the algebra H_Γ^∞. The work in Chapter 4.2 is an attempt to study the matrix-valued interpolation problem in H^∞_Γ when Γ is an amenable group. It is a first attempt to understand the operator algebra structure of $H^\infty_\Gamma/\mathcal{I}$.

Finally, in Chapter 5, we look at the notion of complexity for the matrix-valued interpolation problem via C^*-envelopes.

Chapter 2

Preliminaries

2.1 The spaces $\mathbb{C} + BH^p$

When we refer to the topology on the spaces L^p we will mean the norm topology when $1 \leq p < \infty$ and the weak* topology when $p = \infty$. If $\mathcal{S}$ is a subset of L^p, then we denote by $[\mathcal{S}]_p$ the smallest closed subspace of L^p (weak*-closed, when $p = \infty$) that contains $\mathcal{S}$. For ease of notation, when $p = 2$ we will denote $[\mathcal{S}]_2$ by $[\mathcal{S}]$.

Let $g \in H^\infty$ and let H^p_g denote the space $[\mathbb{C} \cdot 1 + gH^p]_p$. A function $u \in H^p$ is called *outer* if $[H^\infty u]_p = H^p$. If $p = \infty$, then the closure is with respect to the weak* topology. Every function $f \in H^p$ has a factorization of the form $f = \phi u$, where ϕ is inner and u is outer. The inner function ϕ can be factored further into a Blaschke factor B and a singular factor s. The zero set of B is exactly the same as the zero set of f and, the inner function s and the outer function are non-zero on $\mathbb{D}$. The factorization $f = Bsu$ is unique up to multiplication by a unimodular scalar. For proofs of these facts we refer the reader to [22, Chapter 4]. Our first result shows for the space H_g^∞ that it is enough to consider the case where g is an inner function. Note that multiplication by an inner function is isometric on L^p for $1 \leq p \leq \infty$ and also weak* continuous on L^p for $p > 1$.

Proposition 2.1.1. *Let $g \in H^\infty$. If $g = \phi u$ is the factorization of g into its inner part ϕ and outer part u, then we have the following:*

1. *The space H_ϕ^p is a closed subspace of H^p. If $p = \infty$, then H_ϕ^∞ is a weak*-closed subalgebra of H^∞.*

2. *The spaces H_g^p and H_ϕ^p are equal.*

Proof.

1. Since $|\phi| = 1$ a.e. on the circle, we see that ϕH^p is a closed subspace of H^p. It follows quite easily from the fact that the constants are a one-dimensional subspace that $\mathbb{C} + \phi H^p$ is closed. The fact that H_ϕ^∞ is an algebra is trivial.

2. It is straightforward that $\mathbb{C} + gH^p \subseteq H_\phi^p$. For the converse let $\lambda + \phi f \in H_\phi^p$. Since u is outer, by the Helson-Lowdenslager-Srinivasan invariant subspace theorem [21], we may choose a sequence $f_n \in H^p$ (or net $f_t \in H^\infty$, if $p = \infty$) such that $uf_n \to f$ ($uf_t \to f$). It follows that $\lambda + gf_n \to \lambda + \phi f$ and so $H_g^p \supseteq H_\phi^p$.

□

It is well known that the set of zeroes of a function $f \in H^1$ is countable or finite. Further if $\{a_n\}_{n=1}^N$ are the zeroes of f, counting multiplicity, where N is either finite or infinite, then

$$\sum_{n\in\mathbb{N}}(1 - |a_n|) < \infty. \tag{2.1}$$

This condition in (2.1) is known as the *Blaschke condition*. If the Blaschke condition is true, then we can form the *Blaschke product*

$$B(z) = \prod_{n=1}^{N} \frac{|a_n|}{a_n}\left(\frac{a_n - z}{1 - \overline{a_n}z}\right). \tag{2.2}$$

The convergence of the sum in (2.1) guarantees that the Blaschke product B converges and defines a bounded analytic function on $\mathbb{D}$. The boundary values of B are unimodular on the circle and so B is an inner function.

Our primary interest will be in the spaces H^p_B for a Blaschke product B and for $p = 1, 2, \infty$. For the case where B is a Blaschke product we will fix notation as follows. We denote by ϕ_a the elementary Möbius transformation of the disk given by

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z}, \tag{2.3}$$

for $a \neq 0$, and $\phi(z) = z$, when $a = 0$. We will write

$$B = \prod_{j=1}^{N} \frac{|\alpha_j|}{\alpha_j}\phi_{\alpha_j}^{m_j} \tag{2.4}$$

where N is either finite or infinite, $\{\alpha_j\}_{j=1}^{N}$ are distinct and $m_j \geq 1$. The normalizing factor $\dfrac{|\alpha_j|}{\alpha_j}$ is introduced to ensure convergence of the infinite product and is defined to be 1 if $\alpha_j = 0$. We will assume throughout that B has at least 2 zeroes, i.e., $\sum_{j=1}^N m_j \geq 2$. We point out that a function $f \in H^\infty_B$ if and only if it satisfies the following two constraints:

1. $f(\alpha_i) = f(\alpha_j)$, $1 \leq i, j \leq N$.

2. If $m_j \geq 2$, then $f^{(i)}(\alpha_j) = 0$ for $i = 1, \ldots, m_j - 1$.

We will denote by K the Szegö kernel and by k_z the Szegö kernel at the point z, i.e., the element of H^2 such that $f(z) = \langle f, k_z \rangle$ for all $f \in H^2$. Note that the Szegö kernel is actually a bounded analytic function on $\mathbb{D}$ and so is in H^∞. We will use the letter z to represent a complex variable, the identity map on $\mathbb{D}$ and the identity map on $\mathbb{T}$.

2.2 The algebras H^∞_Γ

In addition to the spaces of the form H^∞_B we will be concerned in our work with the fixed point algebra for certain group actions on the disk. In this section we will describe what kinds of groups we are interested in. We will prove some basic facts about the structure of these groups. These results are interesting in their own right. Their main purpose is to enable a discussion of interpolation problems.

An *automorphism* of the disk is a holomorphic map of the disk onto itself, which has a holomorphic inverse. An application of the Schwarz lemma shows that all such maps are of the form

$$z \mapsto \lambda \frac{a - z}{1 - \bar{a}z} = \lambda \phi_a(z), \tag{2.5}$$

where $a \in \mathbb{D}$ and $\lambda \in \mathbb{T}$. The set of automorphisms of the disk becomes a group under the usual composition of maps. All the groups that we will deal with will be groups of automorphisms of the disk.

The group of automorphisms of the disk $\mathbb{D}$ is naturally identified with the group $PSL(2,\mathbb{R})$. The group $PSL(2,\mathbb{R})$ is the quotient of the special linear group $SL(2,\mathbb{R})$ by its center $\{\pm I\}$. Since $SL(2,\mathbb{R})$ is a subgroup of $GL(2,\mathbb{R})$, it is a topological group. A discrete subgroup of $PSL(2,\mathbb{R})$ is called *Fuchsian.* In our work we will be interested primarily in Fuchsian groups.

In Riemann surface theory Fuchsian groups play a central role. We outline the essential facts and direct the reader to the book by Farkas and Kra [17], which contains a thorough discussion of the connections between Fuchsian groups and Riemann surfaces. An excellent introduction to the general theory of Fuchsian groups is Katok [24]. Every Riemann surface R has a universal covering space M and an associated covering map $p : M \to R$. The *universal covering space* is a connected, simply connected Riemann surface and the map p is a covering map. In addition to the usual conditions for a covering map, the map p is also smooth. There are only 3 possible universal covering spaces for a Riemann surface: the disk $\mathbb{D}$, the Riemann sphere S^2 and the complex plane $\mathbb{C}$. The disk is the universal covering space for "most" Riemann surfaces. The exceptions to this are the Riemann sphere S^2, the complex plane $\mathbb{C}$, the two-torus $\mathbb{T}^2$ and the punctured plane $\mathbb{C}^*$.

An automorphism ϕ of the disk is called a *deck transformation* with respect to (M, p) if and only if $p \circ \phi = p$. The set of all deck transformations is a group under the usual composition of maps. When $\mathbb{D}$ is the universal covering space, the group of deck transformations Γ is a torsion-free Fuchsian group that acts without fixed points. The group Γ is also isomorphic to the fundamental group of the Riemann surface R, i.e., $\Gamma \cong \pi_1(R)$. Conversely, if Γ is a discrete, torsion-free group of automorphisms of the disk that acts without fixed points, then the quotient space $\mathbb{D}/\Gamma$ can be given the structure of a Riemann surface and the natural projection $p : \mathbb{D} \to \mathbb{D}/\Gamma$ is the universal covering map. For example, if R is a m-holed region in the complex plane, then $\pi_1(R) = \mathbb{F}_m$, the free group on m generators. In Chapter 4.2 we will see an explicit formula for the covering of an annulus by the disk.

Let $H^\infty(R)$ denote the algebra of bounded analytic functions on the Riemann surface R. The map p induces a map $p^* : H^\infty(R) \to H^\infty$ by composition and the image of p^* is H^∞_Γ, the set of functions in H^∞ that are fixed by the action of Γ. Hence as Banach algebras (operator algebras) $H^\infty(R)$ and H^∞_Γ are isometrically (completely isometrically) isomorphic. We could, in theory, forget the domain R and focus on the group Γ and the space H^∞_Γ. In Chapter 4.2, we will look at the simplest examples of this.

In the Hardy space literature, the identification between $H^\infty(R)$ and H^∞_Γ was originally used to prove results on boundary values of functions in the Hardy spaces of R. Forelli [20] made significant use of this structure in proving the corona theorem for regions in the complex plane. In Chapter 2.3, we will describe Forelli's ideas. His construction provides a way to free ourselves from some of the function theory on multiply connected domains.

If Γ is a Fuchsian group, then we have a natural action of the group Γ on L^∞ given by $f \mapsto f \circ \gamma^{-1}$, where $\gamma \in \Gamma$. This action restricts, since γ is an analytic map, to an automorphism of the space H^∞. The map $f \mapsto f \circ \gamma^{-1}$ also extends to a bounded linear operator on L^1, but is no longer an isometry. Since we will deal with L^p spaces and H^p spaces for different indices p we will denote the corresponding fixed point space for the action of Γ by writing Γ as a subscript. Of primary interest are the cases $p = 1, 2, \infty$.

Abrahamse's results on interpolation show that we need to consider not only the fixed point space but also consider the modulus automorphic functions, i.e., the elements of $H^2_\chi(R)$ [1, 19]. We denote the character group of Γ by $\hat{\Gamma}$. The set $\hat{\Gamma}$ is the set of homomorphisms from Γ into the circle $\mathbb{T}$. Since Γ is discrete these homomorphisms are automatically continuous and $\hat{\Gamma}$ is a compact group under pointwise multiplication when endowed with the topology of pointwise convergence.

Given a character $\chi \in \hat{\Gamma}$ we define the character space L^p_χ as the set of elements in L^p such that $f \circ \gamma = \chi(\gamma) f$ for all $\gamma \in \Gamma$. An element in L^p_χ for some character $\chi \in \hat{\Gamma}$ will be called *character automorphic*. The corresponding Hardy space is defined by $H^p_\chi := L^p_\chi \cap H^p$. We define a function $f \in L^p$ to be *modulus automorphic* if and only if $|f| \in L^P_\Gamma$. The absolute value of a character automorphic element of L^p is modulus automorphic.

We will call two Fuchsian groups Γ_1 and Γ_2 *conjugate* if there exists an automorphism of the disk γ such that $\gamma\Gamma_1 = \Gamma_2\gamma$. If $f \circ \alpha = f$ for all $\alpha \in \Gamma_1$, then $f \circ (\gamma^{-1}\beta\gamma) = f$ for all $\beta \in \Gamma_2$ and so $f \circ \gamma^{-1} \in H^\infty_{\Gamma_2}$. Therefore, the spaces $H^\infty_{\Gamma_1}$ and $H^\infty_{\Gamma_2}$ are completely isometrically isomorphic via the map $f \mapsto f \circ \gamma^{-1}$. Hence, as far as the operator algebra structure is concerned the operator algebra H^∞_Γ is unaffected by conjugating the group Γ.

Two factorization results are central to our work. The first result is the factorization of functions in H^p into their inner and outer factors and the factorization of inner functions into Blaschke products and singular factors. The second result is Riesz factorization, which we stated as Theorem 1.3.2. We will now collect a few elementary facts about the factorization of functions in H^p_χ.

Proposition 2.2.1. *Let Γ be a Fuchsian group and let $\chi \in \hat{\Gamma}$. If $f \in H^p_\chi$, then $|f|$ is modulus automorphic. If $u \in H^p$ is an outer function and $|u| \in L^p_\Gamma$, then there exists $\chi \in \hat{\Gamma}$ such that $u \in H^p_\chi$.*

Proof. If u is outer, then $u \circ \gamma$ is outer for all $\gamma \in \Gamma$, since composition by γ is continuous and invertible on H^p. Two outer function u and v have equal modulus if and only if there exists a scalar $\lambda \in \mathbb{T}$ such that $u = \lambda v$, [16, page 142, Corollary 6.23]. We have $|u \circ \gamma| = |u| \circ \gamma = |u|$ and so there exists $\chi(\gamma) \in \mathbb{T}$ such that $u \circ \gamma = \chi(\gamma)u$. We need to show that $\chi \in \hat{\Gamma}$. If $\gamma_1, \gamma_2 \in \Gamma$, then

$$\begin{aligned} \chi(\gamma_1\gamma_2)u &= u \circ (\gamma_1\gamma_2) && (2.6)\\ &= (u \circ \gamma_1) \circ \gamma_2 && (2.7)\\ &= \chi(\gamma_1)(u \circ \gamma_2) && (2.8)\\ &= \chi(\gamma_1)\chi(\gamma_2)u. && (2.9) \end{aligned}$$

Since u is non-zero, χ is a character. □

Proposition 2.2.2 (Riesz factorization)**.** *Let Γ be a Fuchsian group and let $\chi \in \hat{\Gamma}$. If $f \in H^1_\chi$, then there exists a character $\sigma \in \hat{\Gamma}$, an outer function $u \in H^2_\sigma$ and an inner function $\phi \in H^\infty_{\sigma^{-2}\chi}$ such that $f = \phi u^2$.*

Proof. It is well known [22, Theorem 19] that f has a factorization of the form ϕu^2, where ϕ is inner and $u \in H^2$ is outer. It follows from Proposition 2.2.1

that $|f|^{1/2} = |u| \in L^2_\Gamma$ and so $u \in H^2_\sigma$ for some $\sigma \in \hat{\Gamma}$. It follows easily that $\phi \in H^\infty_{\sigma^{-2}\chi}$. $\square$

Proposition 2.2.3. *If $f \in H^1_\chi$ and $f = Bsu$ is the factorization of f into a Blaschke product B, a singular function s and an outer function u, then there exist characters $\sigma, \theta \in \hat{\Gamma}$ such that $B \in H^\infty_\sigma$, $u \in H^1_\theta$ and $s \in H^\infty_{(\sigma\theta)^{-1}\chi}$.*

Proof. Note that if $f(z) = 0$, then $f(\gamma(z)) = 0$ and so the zeroes of f are made up of the union of countably many disjoint orbits. The Blaschke product B vanishes precisely on the zero set of f. Since γ permutes the orbit of a point, we see that $B \circ \gamma$ also vanishes on the zero set of f and so $B \circ \gamma = BC$ where C is inner. A similar argument shows that $B \circ \gamma^{-1} = BD$ with D an inner function. We have,

$$B = B \circ \gamma \circ \gamma^{-1} \tag{2.10}$$

$$= (BC) \circ \gamma^{-1} \tag{2.11}$$

$$= (B \circ \gamma^{-1})(C \circ \gamma^{-1}) \tag{2.12}$$

$$= (BD)(C \circ \gamma^{-1}) \tag{2.13}$$

$$= B(D)(C \circ \gamma^{-1}). \tag{2.14}$$

Since H^∞ has no zero divisors we get that $D(C\circ\gamma^{-1}) = 1$ and so $\overline{D} = C\circ\gamma^{-1} \in H^\infty$. This shows that D and C are constant and so $B \circ \gamma = \sigma(\gamma)B$ for some scalar $\sigma(\gamma) \in \mathbb{T}$. It is easy to check that $\sigma \in \hat{\Gamma}$.

Since $|f| = |u| \in L^1_\Gamma$, we see by Proposition 2.2.1 that $u \in H^1_\theta$ for some character θ. It now follows from the uniqueness of the factorization that $s \in H^\infty_{(\sigma\theta)^{-1}\chi}$. $\square$

If Γ is a group of automorphisms, then we denote by $\Gamma(z)$ the *orbit* of the point

z under the action of Γ, i.e., $\Gamma(z) := \{\gamma(z) : \gamma \in \Gamma\}$. The *stabilizer subgroup* at the point $z \in \mathbb{D}$ is denoted Γ_z, i.e., $\Gamma_z := \{\gamma \in \Gamma : \gamma(z) = z\}$.

One consequence of the proof Proposition 2.2.3 is the following: if the zero set of the Blaschke product B is the orbit $\Gamma(z)$, then $B \in H^\infty_\sigma$ for some $\sigma \in \hat{\Gamma}$. We will call this character σ the *the character associated to the Blaschke product* B or the *character associated to the the point* z.

Let us describe the connection between the spaces H^∞_Γ and the spaces $H^\infty_{B_z}$, since this motivates the results of Chapter 4.1. Let us assume that Γ is the group of deck transformations that arise from a universal covering map of a bounded multiply connected domain. In this case the connection between the algebras H^∞_B and the algebra H^∞_Γ is the following: The algebra H^∞_Γ is the set of functions in H^∞ that are fixed by Γ. Said differently, this is the set of functions in H^∞ that are constant on the orbits of points under the action of Γ. Since R is assumed to be a bounded domain, we see that the space $H^\infty(R)$ contains a non-constant function. Therefore, for any point $z \in \mathbb{D}$, the points in $\Gamma(z)$ satisfy the Blaschke condition (2.1). If $z \in \mathbb{D}$ and $\Gamma(z)$ is the orbit of z under Γ, then we can form the Blaschke product B_z with zero set $\Gamma(z)$. A function $f \in H^\infty$ is constant on $\Gamma(z)$ if and only if $f \in H^\infty_{B_z} = \mathbb{C} + B_z H^\infty$ and so

$$H^\infty_\Gamma = \bigcap \{H^\infty_{B_z} : z \in \mathbb{D}\}. \tag{2.15}$$

Note in these cases that $\Gamma_z = \{e\}$ for all $z \in \mathbb{D}$.

If we allow our groups to act with fixed points or have torsion, then it is entirely possible that the fixed point space is trivial, i.e., reduces to just the constant function. The simplest example of this is the group generated by an irrational rotation of the disk. Here the group of automorphisms is isomorphic to $\mathbb{Z}$. In this case the Blaschke product B_z does not converge for any $z \neq 0$. Therefore, what we seek is a middle ground.

Since our interest is very much in analytic functions we will be concerned with H^∞_Γ quite a bit. As mentioned in the last paragraph, this space could be trivial. However, when H^∞_Γ is non-trivial, it must be infinite-dimensional. This is a general fact about subalgebras of H^∞.

Proposition 2.2.4. *If $\mathcal{A}$ is a unital subalgebra of H^∞ that contains non-constant functions, then $\mathcal{A}$ is infinite dimensional.*

Proof. Assume to the contrary that $\mathcal{A}$ is n-dimensional. Let $f \in \mathcal{A}$ be non-constant. By subtracting $f(0)$, we may assume that $f(0) = 0$ and $f \neq 0$. The elements $1, f, f^2, \ldots, f^n$ must be linearly dependent and so there exists $a_0, \ldots, a_n$ such that $\sum_{j=0}^n a_j f^j = 0$. Evaluating at $z = 0$, we get $a_0 = 0$ and so $f(a_1 + a_2 f + \ldots + a_n f^{n-1}) = 0$. Since $f \neq 0$, $a_1 + a_2 f + \ldots + a_n f^{n-1} = 0$. Repeating the argument above yields $a_1 = \ldots = a_n = 0$, a contradiction. $\square$

Proposition 2.2.5 and Proposition 2.2.6 provide some consequences about the structure of Γ that follow from assuming that the space H^∞_Γ is non-trivial. These results allow us to deal with torsion and fixed points in a systematic way. The result also explains why it is enough to work with Fuchsian groups acting on the disk.

Proposition 2.2.5. *Let Γ be a group of automorphisms and let Γ_0 be the stabilizer at the origin. If Γ_0 is finite, then Γ_0 is cyclic.*

Proof. Let $\gamma \in \Gamma_0$ and let $m = |\Gamma_0|$. We have $\gamma(0) = 0$ and so there exists a constant $\lambda \in \mathbb{T}$ such that $\gamma(z) = \lambda z$. Note that $\lambda^m = 1$ and so Γ_0 is a subgroup of the cyclic group $\langle \rho \rangle$, where ρ is rotation through the angle $\dfrac{2\pi}{m}$. Since Γ_0 is a subgroup of a cyclic group, the group Γ_0 is cyclic and $\Gamma_0 = \langle \rho \rangle$, since both Γ_0 and ρ have order m. □

Proposition 2.2.6. *Let Γ be a group of automorphisms of the disk. Assume that the algebra H^∞_Γ is non-trivial, i.e., contains a non-constant function. The following are true:*

1. *The stabilizer Γ_0 is a finite, cyclic group.*
2. *The infinite Blaschke sum $\sum_{\gamma \in \Gamma}(1 - |\gamma(0)|)$ converges.*
3. *The group Γ is Fuchsian (discrete).*

Proof.

1. Every element $\gamma \in \Gamma_0$ fixes the origin. Hence, γ is a rotation of the disk and there exists a constant $\lambda \in \mathbb{T}$ such that $\gamma(z) = \lambda z$. Let f be a non-constant function in H^∞_Γ and let a_k, $k \neq 0$, be a non-zero Fourier coefficient of f. For every $\gamma \in \Gamma_0$ we have $f(\gamma(z)) = f(z)$ and so $\lambda^k a_k = a_k$. This yields, $\lambda^k = 1$ and so Γ_0 is finite. The claim about Γ_0 being cyclic follows from Proposition 2.2.5.

2. If $f \in H^\infty_\Gamma$ is non-constant, then by subtracting a constant we may assume that f is non-zero and vanishes at 0. Since $f \circ \gamma = f$ we see that $f(\gamma(0)) = f(0) = 0$ for all $\gamma \in \Gamma$ and so $\sum_{w \in \Gamma(0)}(1 - |w|) < \infty$. If $\alpha \in \Gamma$, then

the cardinality of Γ_0 is equal to the cardinality of $\Gamma_{\alpha(0)}$. In fact, the two stabilizers are isomorphic via the map $\gamma \mapsto \alpha\gamma\alpha^{-1}$. Therefore,

$$\sum_{\gamma\in\Gamma}(1-|\gamma(0)|) = |\Gamma_0| \sum_{w\in\Gamma(0)}(1-|w|) < \infty. \tag{2.16}$$

3. If Γ is not discrete, then there exists a sequence of distinct elements $\gamma_n \in \Gamma$ such that $\gamma_n \to e$, the identify element of Γ. This forces $\gamma_n(0) \to 0$ which contradicts the convergence of the Blaschke sum.

□

Note that if the series $\sum_{\gamma\in\Gamma}(1-|\gamma(0)|)$ does converge, then the argument above shows that Γ_0 is finite and Γ is discrete.

If the stabilizer Γ_0 is finite, and the series $\sum_{\gamma\in\Gamma}(1-|\gamma(0)|)$ converges, then we define two Blaschke products B_0 and B that arise naturally. First consider the Blaschke product B_0 whose zero set is $\Gamma(0)$. We call B_0 the *Blaschke product for the orbit* $\Gamma(0)$. Let $m = |\Gamma_0|$ and define $B := B_0^m$. We call B the *Blaschke product associated to* Γ. If it is the case that only the identity map fixes the origin, then $m = 1$ and $B = B_0$. If $\gamma \in \Gamma$, then $B_0 \circ \gamma$ is a Blaschke product whose zero set is the same as the zero set of B_0. It follows, just as in Proposition 2.2.3, that $B_0 \circ \gamma = \chi(\gamma)B_0$ for some character χ.

Let $\gamma \in \Gamma_0$. Since $\gamma(0) = 0$, there exists a scalar $\lambda \in \mathbb{T}$ such that $\gamma(z) = \lambda z$. Let $\gamma_n(0)$, for $n = 1, \ldots, N$, be an enumeration of the distinct non-zero elements of $\Gamma(0)$. Here N can be either finite or infinite. The Blaschke product B_0 can be written

$$B_0(z) = z\prod_{n=1}^{N} \frac{|\gamma_n(0)|}{\gamma_n(0)} \frac{\gamma_n(0) - z}{1 - \overline{\gamma_n(0)}z}. \tag{2.17}$$

Hence,

$$B_0(\lambda z) = (\lambda z)\prod_{n=1}^{N} \frac{|\gamma_n(0)|}{\gamma_n(0)} \frac{\gamma_n(0) - \lambda z}{1 - \overline{\gamma_n(0)}\lambda z} \tag{2.18}$$

$$= \lambda z\prod_{n=1}^{N} \frac{|\overline{\lambda}\gamma_n(0)|}{\overline{\lambda}\gamma_n(0)} \frac{\overline{\lambda}\gamma_n(0) - z}{1 - \overline{\gamma_n(0)}\lambda z}. \tag{2.19}$$

We have, $\overline{\lambda}\gamma_n(0) = \gamma^{-1}(\gamma_n(0))$ and so $\overline{\lambda}\gamma_n(0)$, for $n = 1, \dots, N$, is another enumeration of the non-zero points in the orbit $\Gamma(0)$. Hence, $B_0(\lambda z) = \lambda B_0(z)$. If $|\Gamma_0| = m$, then $\gamma(z) = e^{\frac{2\pi i}{m}}(z)$ is a generator for Γ_0 and so $B_0(e^{\frac{2\pi i}{m}}z) = e^{\frac{2\pi i}{m}}B_0(z)$ for all $z \in \mathbb{D}$.

Proposition 2.2.8 will show that we can build an orthonormal basis from the Blaschke product B associated with the group Γ. The Hilbert spaces H^2_χ, for $\chi \in \hat{\Gamma}$, are subspaces of H^2. Therefore, H^2_χ is a reproducing kernel Hilbert space with kernel function K^χ. We denote by k^χ the normalized kernel function for the space H^2_χ at the point 0. For the space H^2_Γ this is just the constant function 1.

Lemma 2.2.7. *Let Γ be a Fuchsian group, let B_0 be the Blaschke product for the orbit of the origin, let $m = |\Gamma_0|$ and let $B = B_0^m$ be the Blaschke product associated with the group Γ. Let χ be the character such that $B_0 \in H^\infty_\chi$. If $f \in H^2_\Gamma$, then $B|(f - f(0))$.*

Proof. Let $\gamma \in \Gamma_0$ be a generator with $\gamma(z) = \lambda z = e^{\frac{2\pi i}{m}}z$. If $f \circ \gamma = f$, then $f(\lambda z) = f(z)$ for all $z \in \mathbb{D}$. Let a_k denote the k^{th} Fourier coefficient of f. We have $\lambda^k a_k = a_k$ for all $k \geq 0$. Since λ has order m, we see that $a_k = 0$ unless $m|k$.

Therefore, $g := f - f(0) = z^m h$ for some function $h \in H^2$. Since $g \in H^2_\Gamma$ and 0 is a root of g of multiplicity m we see that every point in the orbit $\Gamma(0)$ is a zero of multiplicity m for g. Hence, $B_0^m = B$ divides $g = f - f(0)$. $\square$

If $f(0) = 0$ and we write $f = Bh$, then we see that

$$B(z)h(z) = B(\lambda z)h(\lambda z) = \lambda^m B(z)h(\lambda z). \tag{2.20}$$

Since $|\Gamma_0| = m$, we get $h(\lambda z) = h(z)$. Therefore, if $h(0) = 0$, then we can repeat the argument from the above proof to show that $B|h$.

Proposition 2.2.8. *Let Γ be a Fuchsian group, let B_0 be the Blaschke product for the orbit of the origin, let $m = |\Gamma_0|$ and let $B = B_0^m$ be the Blaschke product associated with the group Γ. Let χ be the character such that $B \in H^\infty_\chi$. An orthonormal basis $\mathcal{E}$ of H^2_Γ is given by the non-zero elements of the set $\{B^n k^{\overline{\chi}^n} :$ $n \geq 0\}$.*

Proof. It is straightforward to check that $\mathcal{E}$ is orthonormal. Suppose that $f \in H^2_\Gamma$, $f \neq 0$ and $f \perp \mathcal{E}$. Since $f \perp 1$ we have $f(0) = 0$ and $f = Bf_1$. Since $f \circ \gamma = f$ for all $\gamma \in \Gamma$, we see that f vanishes on the orbit of 0 and so $f = Bf_1$, by Lemma 2.2.7. Now, $Bf_1 = f = f \circ \gamma = (B \circ \gamma)(f_1 \circ \gamma) = \chi(\gamma) Bf_1 \circ \gamma$ and so $f_1 \circ \gamma = \overline{\chi}(\gamma) f_1$. Now, $0 = \left\langle Bf_1, Bk^{\chi^{-1}} \right\rangle = \left\langle f_1, k^{\chi^{-1}} \right\rangle = f_1(0)$. Repeating the argument above we see that $f = B^n f_n$ with $f_n(0) = 0$. Since f is non-zero , f can only have a zero of finite multiplicity at the origin, a contradiction. $\square$

Corollary 2.2.9. *Let Γ be a Fuchsian group, let B_0 be the Blaschke product for the orbit of the origin, let $m = |\Gamma_0|$ and let $B = B_0^m$ be the Blaschke product* *associated with the group* Γ*. If the Blaschke product* B *is invariant under* Γ*, then* $\{B^n : n \geq 0\}$ *is an orthonormal basis for* H^2_Γ*.*

2.3 The Forelli projection

To begin this section we describe a construction due to Forelli [20].

Let Γ be a Fuchsian group and let m denote normalized Lebesgue measure on the circle $\mathbb{T}$. A set $E \subseteq \mathbb{T}$ is called Γ-*invariant* if and only if $m(E \triangle \gamma^{-1}(E)) = 0$ for all $\gamma \in \Gamma$. The set of all measurable Γ-invariant sets forms a sub σ-algebra of the algebra of Lebesgue measurable sets and we denote this sub σ-algebra $\mathcal{M}_\Gamma$. The set E is Γ-invariant if and only if $\chi_{\gamma^{-1}(E)} = \chi_E$ a.e. with respect to m, for all $\gamma \in \Gamma$. The characteristic functions of Γ-invariant sets generate the algebra $L^\infty(\mathbb{T}, \mathcal{M}_\Gamma, m)$. The algebra L^∞_Γ is generated by its projections, i.e., by the characteristic functions that satisfy $\chi_{\gamma^{-1}(E)} = \chi_E \circ \gamma = \chi_E$ for all $\gamma \in \Gamma$. This shows that we have a natural identification between L^∞_Γ and the space $L^\infty(\mathbb{T}, \mathcal{M}_\Gamma, m)$. A similar argument identifies the corresponding L^p spaces.

Given an element $g \in L^p$, $p \geq 1$, consider the linear functional $l_g : L^q_\Gamma \to \mathbb{C}$ defined by

$$l_g(f) := \int_{\mathbb{T}} f g \, dm. \tag{2.21}$$

This functional is weak* continuous on L^∞. By a standard duality argument l_g is induced by integration against an element $\tilde{g} \in L^q_\Gamma$. If we let $\Phi : L^p \to L^p_\Gamma$ be defined by $\Phi(g) = \tilde{g}$, then Φ is the usual conditional expectation of L^p onto $L^p(\mathbb{T}, \mathcal{M}_\Gamma, m)$ that arises in probability theory. The following properties are well known.

1. The map Φ is a projection, i.e., $\Phi^2 = \Phi$.

2. When $p = 2$, the projection Φ is selfadjoint.

3. For $p > 1$, the projection Φ is weak* continuous.

4. For $p \geq 1$, $f \in L^p$ and $E \in \mathcal{M}_\Gamma$

$$\int_E \Phi(f) = \int_E f. \tag{2.22}$$

5. For $1 \leq p \leq \infty$, q such that $q^{-1} + p^{-1} = 1$, $f \in L^p$ and $g \in L^q_\Gamma$,

$$\Phi(fg) = \Phi(f)g \tag{2.23}$$

If we combine the properties in (2.22) and (2.23) we also get

$$\int_E \Phi(f)g = \int_E fg = \int_E \Phi(fg), \tag{2.24}$$

where $E \in \mathcal{M}_\Gamma$.

The corresponding Hardy space is defined by $H^p_\Gamma := H^p \cap L^1_\Gamma$. In the classical case the Lebesgue space L^2 is the orthogonal direct sum of H^2 and $\overline{H^2_0}$, i.e., every square integrable function on the circle is the sum of an analytic and an anti-analytic part. For the spaces L^2_Γ this is no longer true.

Let $f \in L^2$ and decompose $f = g + \overline{h}$, where $g \in H^2$ and $h \in H^2_0$. Composing with $\gamma \in \Gamma$ we get

$$f = f \circ \gamma = g \circ \gamma + \overline{h \circ \gamma} = g + \overline{h}. \tag{2.25}$$

From this we get $g \circ \gamma - g = \overline{h - h \circ \gamma} \in \mathbb{C}$ and so $g \circ \gamma - g \equiv \overline{h - h \circ \gamma} \equiv c_\gamma$, where c_γ is a constant (that depends on f). The constant is easily computed by integrating

$$c_\gamma = \int g \circ \gamma - g \tag{2.26}$$

$$= g(\gamma(0)) - g(0) \tag{2.27}$$

$$= \left\langle g, k_{\gamma(0)} - 1 \right\rangle . \tag{2.28}$$

Similarly, $c_\gamma = \left\langle \overline{h}, 1 - \overline{k_{\gamma(0)}} \right\rangle$. Combining these two equations, we get

$$2c_\gamma = \left\langle f, k_{\gamma(0)} - \overline{k_{\gamma(0)}} \right\rangle \tag{2.29}$$

$$= \left\langle \Phi(f), k_{\gamma(0)} - \overline{k_{\gamma(0)}} \right\rangle \tag{2.30}$$

$$= \left\langle f, \Phi(k_{\gamma(0)} - \overline{k_{\gamma(0)}}) \right\rangle . \tag{2.31}$$

If we set $v_\gamma = -i\Phi(k_{\gamma(0)} - \overline{k_{\gamma(0)}})$, then v_γ is a real-valued function in L^∞_Γ with the property that

$$\int f v_\alpha = 2ic_\gamma \tag{2.32}$$

for all $\gamma \in \Gamma$. The *defect space* N is defined by

$$N := \operatorname{span}\{v_\gamma \,:\, \gamma \in \Gamma\}. \tag{2.33}$$

If $f \in L^2_\Gamma$ is orthogonal to v_γ for all $\gamma \in \Gamma$, then $g \circ \gamma - g = 0$ and so $g \in H^2_\Gamma$. Similarly, $h \in H^2_{0,\Gamma}$ and we see that $L^2_\Gamma = H^2_\Gamma \oplus \overline{H^2_{0,\Gamma}} \oplus [N]$.

If $\gamma_1, \gamma_2 \in \Gamma$, then

$$c_{\gamma_1 \circ \gamma_2} = g \circ (\gamma_1 \circ \gamma_2) - g \tag{2.34}$$

$$= g \circ (\gamma_1 \circ \gamma_2) - g \circ \gamma_2 + g \circ \gamma_2 - g \tag{2.35}$$

$$= (g \circ \gamma_1 - g) \circ \gamma_2 + (g \circ \gamma_2 - g) \tag{2.36}$$

$$= c_{\gamma_1} + c_{\gamma_2} \tag{2.37}$$

It follows that

$$\int f v_{\gamma_1 \circ \gamma_2} = 2ic_{\gamma_1 \circ \gamma_2} = 2i(c_{\gamma_1} + c_{\gamma_2}) = \int f(v_{\gamma_1} + v_{\gamma_2}). \tag{2.38}$$

Since this is true for all $f \in L^2_\Gamma$ we get $v_{\gamma_1 \circ \gamma_2} = v_{\gamma_1} + v_{\gamma_2}$.

What all of this shows is that the map $\gamma \mapsto v_\gamma$ is a homomorphism from Γ into the additive group N. This homomorphism must factor through the commutator subgroup $[\Gamma, \Gamma]$ to give a homomorphism from $\Gamma/[\Gamma, \Gamma]$ into N. Given a set of generators $\{\gamma_s : s \in S\} \subseteq \Gamma/[\Gamma, \Gamma]$ the vectors $v_s := v_{\gamma_s}$, for $s \in S$, span the space N. If the group Γ is finitely generated, then the space N is finite dimensional and the dimension of N is smaller than the minimal number of generators of $\Gamma/[\Gamma, \Gamma]$. Forelli showed that the dimension of N is equal to the minimal number of generators of $\Gamma/[\Gamma, \Gamma]$ in the case where Γ is the group of deck transformations associated with a universal covering map of a multiply connected domain. As shown in [20] many results follow from this equality, including a corona theorem for H^∞_Γ, the key stepping stone being the construction of a bounded projection $P : H^\infty \to H^\infty_\Gamma$. Since we are not assuming the stronger condition that our group Γ arises as a group of deck transformations, we cannot use this latter projection.

If an element $\gamma \in \Gamma$ has finite order, say m, then $mv_\gamma = v_{\gamma^m} = 0$ and so $v_\gamma = 0$. If $\Gamma/[\Gamma, \Gamma]$ is generated by elements of finite order, then N is trivial. This is both useful and important. It will show in particular that a very naive generalization of the result on C^*-envelopes from [28] is false.

We will assume in all our work that N is finite-dimensional. This is a natural condition from a function theory point of view. The assumption about N means that $H^p_\Gamma + N$ is a closed subspace of L^p for all $1 \leq p \leq \infty$. It is useful to keep in mind that $N \subset L^\infty_\Gamma \subseteq L^p_\Gamma$ for all $1 \leq p \leq \infty$. Note that when $q > 1$, the spaces H^q_Γ and $H^q_\Gamma + N$ are also weak* closed. This is a simple consequence of the fact that weak* limits preserve point values and the fact that N is assumed finite-dimensional. Since duality arguments will play a central role in our interpolation results we would like to gather some results on the duality between the different H^p_Γ spaces.

Proposition 2.3.1. *For $1 \leq p < \infty$, the dual of L^p_Γ can be identified with L^q_Γ, where $q = \frac{p}{p-1}$. In this identification the following are true:*

1. $(H^p_\Gamma)^\perp = H^q_{0,\Gamma} + N$
2. $(H^q_\Gamma)_\perp = H^p_{0,\Gamma} + N$
3. $(H^q_\Gamma + N)_\perp = H^p_{0,\Gamma}$
4. $(H^p_\Gamma + N)^\perp = H^q_{0,\Gamma}$

Proof. The statement about L^p_Γ spaces follows from standard facts about the L^p spaces of a probability measure.

For $p = 2$, we have seen already that $L^2_\Gamma = H^2_\Gamma \oplus \overline{H^2_{0,\Gamma}} \oplus N$ and so the results above are valid. It is easy to see that

$$\int \Phi(h)g = \int hg = 0, \tag{2.39}$$

for all $h \in H^2$, $g \in H^2_{0,\Gamma}$ and so $\Phi(H^2) = L^2_\Gamma \ominus \overline{H^2_{0,\Gamma}} = H^2_\Gamma + N$.

We will show that $H^p_\Gamma = (H^q_{0,\Gamma} + N)_\perp$ and the remaining results will follow either by duality or by a similar argument. Note that the spaces H^q_Γ and $H^q + N$ are weak* closed in L^q.

We first show that $\Phi(H^p) = H^p_\Gamma + N$. Let $f \in L^q_\Gamma$. Note that $f \in \Phi(H^p)^\perp$ if and only if $\int f\Phi(g) = \int fg = 0$ for all $g \in H^p$ if and only if $f \in H^q_0 \cap L^q_\Gamma = H^q_{0,\Gamma}$. Hence, $\Phi(H^p)^\perp = H^q_{0,\Gamma}$.

Let $h \in H^p_\Gamma$, $v \in N$ and $g \in H^q_{0,\Gamma}$. Since $v_\gamma \in L^\infty_\Gamma$ and g is analytic we get

$$\int gv_\gamma = \int g\Phi(k_{\gamma(0)} - \overline{k_{\gamma(0)}}) \tag{2.40}$$

$$= \int g(k_{\gamma(0)} - \overline{k_{\gamma(0)}}) \tag{2.41}$$

$$= g(\gamma(0)) - g(0) = 0. \tag{2.42}$$

It follows that

$$\int g(h+v) = \int gh + \int gv = 0, \tag{2.43}$$

which gives $H^p_\Gamma + N \subseteq (H^q_{0,\Gamma})_\perp$. This also yields

$$\Phi(H^p) = (\Phi(H^p)^\perp)_\perp = (H^q_{0,\Gamma})_\perp \supseteq H^p_\Gamma + N. \tag{2.44}$$

Let $g \in (H^p_\Gamma + N)^\perp$ and let $f \in H^p$. We will show that

$$\int gf = \int g\Phi(f) = 0, \tag{2.45}$$

which will establish the fact that $(H^p_\Gamma + N)^\perp \subseteq \Phi(H^p)^\perp$. We know that $\int g(h+v) = 0$ for all $h \in H^p_\Gamma$ and $v \in N$. Let $f_n \in H^\infty$ and suppose that $f_n \to f$ in the p-norm. Since $f_n \in H^2$, we can write $\Phi(f_n) = h_n + v_n$, where $h_n \in H^2_\Gamma$ and $v_n \in N$. However, $v_n \in L^\infty$ and so $h_n \in H^\infty_\Gamma \subseteq H^p_\Gamma$. Therefore,

$$\int g(h_n + v_n) = \int g\Phi(f_n) = 0 \tag{2.46}$$

for all n. Since $g \in \Phi(H^p)^\perp$ and

$$\int g\Phi(f) = \lim_{n\to\infty} \int g\Phi(f_n) = 0. \tag{2.47}$$

We have established that $\Phi(H^p)^\perp = H^q_{0,\Gamma}$ and that $\Phi(H^p)^\perp \supseteq (H^p_\Gamma + N)^\perp$. This combined with the fact that $\Phi(H^p) \supseteq H^p_\Gamma + N$ shows us that $\Phi(H^p) = H^p_\Gamma + N$ and $(H^p_\Gamma + N)^\perp = H^q_{0,\Gamma}$. □

Proposition 2.3.2. *For $1 \leq p < \infty$, the closure $[H^\infty_\Gamma]_p = H^p_\Gamma$.*

Proof. Consider the case $p \geq 2$. Let $f \in H^p_\Gamma$ and let $f_n \in H^\infty$ converge to f in L^p. Since the L^p norm dominates the L^2 norm we see that $f_n \to f$ in L^2. If we project f_n onto $\Phi(f_n)$ we can choose $g_n \in H^\infty_\Gamma$ and $v_n \in N$ such that $g_n + v_n = \Phi(f_n)$ and $g_n + v_n \to f$ in the L^p norm and in the L^2 norm. In particular $\|v_n\|_2 \to 0$. However, on the finite-dimensional space N the L^2 norm and the L^p norm are equivalent and so $\|v_n\|_p \to 0$. It follows that $g_n \to f$ in L^p.

Let $1 \leq p < 2$, let $H_\Gamma^\infty \subseteq H_\Gamma^p$ and let $(H_\Gamma^\infty)^\perp$ denote the annihilator of H_Γ^∞ in L_Γ^q, where $2 \leq q$ and $q^{-1} + p^{-1} = 1$. We have

$$\begin{aligned}
(H_\Gamma^\infty)^\perp &= \left\{ f \in L_\Gamma^q \, : \, \int fg = 0 \text{ for all } g \in H_\Gamma^\infty \right\} &(2.48)\\
&\subseteq \left\{ f \in L_\Gamma^2 \, : \, \int fg = 0 \text{ for all } g \in H_\Gamma^\infty \right\} &(2.49)\\
&= L_\Gamma^2 \ominus \left[\overline{H_\Gamma^\infty}\right] = L_\Gamma^2 \ominus \overline{H_\Gamma^2} = H_{0,\Gamma}^2 + N. &(2.50)
\end{aligned}$$

Hence, $(H_\Gamma^\infty)^\perp \subseteq L_\Gamma^q \cap (H_{0,\Gamma}^2 + N) = H_{0,\Gamma}^q + N$, since $N \subseteq L_\Gamma^\infty \subseteq L_\Gamma^q$. The reverse inequality follows from $H_{0,\Gamma}^q + N \subseteq H_{0,\Gamma}^2 + N$, $H_\Gamma^\infty \subseteq H_\Gamma^2$ and the result for $p = 2$. We have shown that $(H_\Gamma^\infty)^\perp = H_{0,\Gamma}^q + N$. It follows from Proposition 2.3.1 that $[H_\Gamma^\infty]_p = ((H_\Gamma^\infty)^\perp)_\perp = (H_{0,\Gamma}^q + N)_\perp = H_\Gamma^p$. $\square$

Chapter 3

Distance Formulae

3.1 Invariant subspaces and reflexivity

While discussing Sarason's approach to the Nevanlinna-Pick problem in Chapter 1.2, we saw that there is a close connection between the Nevanlinna-Pick theorem and shift invariant subspaces of H^2. Motivated by these connections, and in preparation for later results, we begin by classifying the subspaces of L^p that are invariant for the algebra H^∞_B. This generalizes the Helson-Lowdenslager-Srinivasan theorem.

Theorem 3.1.1. *Let B be an inner function and let $\mathcal{M}$ be a closed subspace of L^p which is invariant for H^∞_B. Either there exists a measurable set E such that $\mathcal{M} = \chi_E L^p$ or there exists a unimodular function ϕ such that $\phi B H^p \subseteq \mathcal{M} \subseteq \phi H^p$. In the latter case, if $p = 2$, then there exists a subspace $W \subseteq H^2 \ominus BH^2$ such that $\mathcal{M} = \phi(W \oplus BH^2)$.*

Proof. The space $[BH^\infty \mathcal{M}]_p$ is an H^∞-invariant, closed subspace of L^p and since

B is inner $[BH^\infty\mathcal{M}]_p = B[H^\infty\mathcal{M}]_p$. By the invariant subspace theorem for H^∞, either $[H^\infty\mathcal{M}]_p = \chi_E L^p$ for some measurable subset E of the circle or $[H^\infty\mathcal{M}]_p = \phi H^p$ for some unimodular function ϕ. In the former case

$$\mathcal{M} \supseteq B[H^\infty\mathcal{M}]_p = B\chi_E L^p = \chi_E L^p \supseteq \mathcal{M}. \tag{3.1}$$

In the latter case we see that $\phi H^p = [H^\infty\mathcal{M}] \supseteq \mathcal{M} \supseteq B[H^\infty\mathcal{M}]_p = \phi B H^p$. For the case $p = 2$ since $BH^2 \subseteq \overline{\phi}\mathcal{M} \subseteq H^2$ we see that $\mathcal{M} = \phi(W \oplus BH^2)$ where $W \subseteq H^2 \ominus BH^2$. $\square$

As a corollary we obtain:

Corollary 3.1.2 ([15, Theorem 2.1]). *Let H_1^∞ denote the algebra of functions in H^∞ such that $f'(0) = 0$. A subspace $\mathcal{M}$ of L^2 is invariant for H_1^∞, but not invariant for H^∞, if and only if there exists a unimodular function ϕ, scalars $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$, $\alpha \neq 0$, such that $\mathcal{M} = \phi([\alpha + \beta z] \oplus z^2 H^2)$.*

Proof. From the previous result we see that $\mathcal{M} = \phi(W \oplus z^2H^2)$ where $W \subseteq H^2 \ominus z^2H^2 = \operatorname{span}\{1, z\}$. Since $\mathcal{M}$ is not invariant for H^∞ see that W is one-dimensional and $\alpha \neq 0$. $\square$

We will identify unimodular functions that differ only by a constant factor of modulus 1. If $\mathcal{S}$ is a subspace of H^2, then Beurling's theorem tells us that $[H^\infty\mathcal{S}] = \phi H^2$ for some inner function $\phi_\mathcal{S}$. The inner function $\phi_\mathcal{S}$ is called the *inner divisor* of $\mathcal{S}$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two subsets of H^2, then we define their *greatest common divisor* $\gcd(\mathcal{S}_1, \mathcal{S}_2)$ to be the inner divisor of $[H^\infty(\mathcal{S}_1 \cup \mathcal{S}_2)]$ and the *least common multiple* $\operatorname{lcm}(\mathcal{S}_1, \mathcal{S}_2)$ to be the inner divisor of $[H^\infty\mathcal{S}_1] \cap [H^\infty\mathcal{S}_1]$. For a function f

the inner divisor of $\{f\}$ is clearly the inner factor of f. For functions $f_1, f_2 \in H^2$ we define $\gcd(f_1, f_2) := \gcd(\{f_1\}, \{f_2\})$ and $\operatorname{lcm}(f_1, f_2) := \operatorname{lcm}(\{f_1\}, \{f_2\})$. For a more detailed description of these operations, we refer the reader to [13].

Let $\mathcal{A} \subseteq B(\mathcal{H})$ be an operator algebra. Associated to this operator algebra is its *lattice of invariant subspaces*, which is defined as the set of subspaces of $\mathcal{H}$ that are invariant for $\mathcal{A}$. We will denote the lattice of non-trivial, invariant subspaces of $\mathcal{A}$ by $\operatorname{Lat}(\mathcal{A})$.

An important consequence of Beurling's theorem is that it allows a complete description of the lattice of invariant subspaces for H^∞. Two shift invariant subspaces ϕH^2 and ψH^2 are the equal if and only if $\phi = \lambda\psi$ for a unimodular constant λ. Since we have chosen to identify inner functions that differ only by a constant, we see that that the shift invariant subspaces of H^2 are parameterized by inner functions. There is a natural ordering of inner functions. If ϕ, ψ are inner functions, then we say that $\phi \leq \psi$ if and only if there exists an inner function θ such that $\phi\theta = \psi$. This makes the set of inner functions a lattice with meet and join given by

$$\phi \wedge \psi = \gcd(\phi, \psi), \phi \vee \psi = \operatorname{lcm}(\phi, \psi). \tag{3.2}$$

In this ordering the inner function 1 is the least element of the lattice and the lattice has no upper bound. The map $\phi \mapsto \phi H^2$ is a bijection between the lattice of inner functions and the the lattice of non-trivial, invariant subspaces for H^∞. This identification is a lattice anti-isomorphism, i.e., order reversing isomorphism, taking meets to joins and joins to meets.

For the lattice $\operatorname{Lat}(H_B^\infty)$ the situation is different. There are two parameters that determine an invariant subspace $\mathcal{M} \in \mathrm{Lat}(H^\infty_B)$, an inner function ϕ and a subspace $W \subseteq H^2 \ominus BH^2$. However, the subspace $\mathcal{M}$ does not uniquely determine ϕ and W. Conversely, different choices of ϕ and W can sometimes give rise to the same subspace. A simple example is obtained by setting $B = z^2$, in which case

$$zH^2 = z\left([1, z] \oplus z^2H^2\right) = [z] \oplus z^2H^2. \tag{3.3}$$

If $\mathcal{M} = \phi(W \oplus BH^2)$, then the subspace $W = \overline{\phi}\mathcal{M} \ominus BH^2$. It is always possible to make a canonical choice of inner function and subspace W. The canonical choice is to set the inner function equal to $\phi_\mathcal{M}$, the inner divisor of $\mathcal{M}$, and to let $W_\mathcal{M} = \overline{\phi_\mathcal{M}}\mathcal{M} \ominus BH^2$.

We now describe the extent to which the decomposition of the subspace $\mathcal{M}$ into the form $\phi(W \oplus BH^2)$ fails to be unique. It is useful to keep in mind the rather trivial example in (3.3). Note that in addition to being $H^\infty_{z^2}$-invariant the subspace zH^2 is also shift invariant.

Theorem 3.1.3. *Let $\mathcal{M} \in \mathrm{Lat}(H^\infty_B)$, let $\phi_\mathcal{M}$ be the inner divisor of $\mathcal{M}$ and let $W_\mathcal{M} = \overline{\phi_\mathcal{M}}\mathcal{M} \ominus BH^2$. Let ψ be inner and V be a subspace of $H^2 \ominus BH^2$ such that $\mathcal{M} = \psi(V \oplus BH^2)$. The following are true:*

1. *The inner function* $\gcd(W_\mathcal{M}, B) = 1$.

2. *The inner function $\psi \leq \phi_\mathcal{M}$ and*

$$\phi_\mathcal{M} W_\mathcal{M} = \psi V \oplus B(\psi H^2 \ominus \phi_\mathcal{M} H^2). \tag{3.4}$$

3. *If θ is such that $\psi\theta = \phi_\mathcal{M}$, then $\theta = gcd(B, V)$.*

4. *We have $\phi_{\mathcal{M}} = \psi$ if and only if $W_{\mathcal{M}} = V$.*

5. *If $\mathcal{M} \notin \mathrm{Lat}(H^\infty_C)$ for all $C < B$, then $\psi = \phi_{\mathcal{M}}$ and $V = W_{\mathcal{M}}$.*

Proof.

1. Note that $\phi_{\mathcal{M}} \gcd(W_{\mathcal{M}}, B)$ is an inner function that divides $\mathcal{M}$. Since $\phi_{\mathcal{M}}$ is the inner divisor of $\mathcal{M}$ we get $\gcd(W_{\mathcal{M}}, B) = 1$.

2. Since ϕ is the inner divisor of $\mathcal{M}$, it follows that $\psi|\phi$. Let θ be the inner function such that $\psi\theta = \phi$. We have,

$$\psi\theta(W_{\mathcal{M}} \oplus BH^2) = \phi_{\mathcal{M}}(W_{\mathcal{M}} \oplus BH^2) = \psi(V \oplus BH^2). \tag{3.5}$$

   It follows that

$$\theta W_{\mathcal{M}} \oplus \theta BH^2 = V \oplus BH^2 = V \oplus B(H^2 \ominus \theta H^2) \oplus B\theta H^2. \tag{3.6}$$

   Hence,

$$\theta W_{\mathcal{M}} = V \oplus B(H^2 \ominus \theta H^2). \tag{3.7}$$

   Multiplying by ψ gives (3.4).

3. From (3.7) we see that θ divides both V and B and so $\theta \leq \gcd(B, V)$. From (3.7) we get that $\gcd(B, V)|\theta W_{\mathcal{M}}$. Since $\gcd(W_{\mathcal{M}}, B) = 1$ it must be the case that $\gcd(B, V) \leq \theta$. Hence, $\theta = \gcd(B, V)$.

4. The conditions $\psi = \phi_{\mathcal{M}}$ and $W_{\mathcal{M}} = V$ are equivalent. If $\psi = \phi_{\mathcal{M}}$, then (3.7) shows that $W_{\mathcal{M}} = V$. Conversely, if $W_{\mathcal{M}} = V$, then (3.7) shows that $\theta W_{\mathcal{M}} \supseteq$

$W_{\mathcal{M}}$. If $w \in W_{\mathcal{M}} \subseteq \theta W_{\mathcal{M}}$, then there exists $w_1 \in W_{\mathcal{M}}$ such that $w = \theta w_1$. Repeating the argument we find that there exists $w_n \in W_{\mathcal{M}}$ such that $\theta^n w_n = w$. If $\theta \neq 1$, then the equation $\theta^n w_n = w$ for all $n \geq 0$, contradicts the fact that θ cannot divide w with infinite multiplicity. Hence, $\theta = 1$ and $\phi_{\mathcal{M}} = \psi\theta = \psi$.

5. If $\theta \neq 1$, then

$$\mathcal{M} = \psi(V \oplus BH^2) = \psi\theta(X \oplus CH^2), \tag{3.8}$$

where $C < B$. Hence, $\mathcal{M} \in \mathrm{Lat}(H^\infty_C)$.

□

Theorem 3.1.3 indicates that the lattice of invariant subspaces for H^∞_B is more complex than the lattice of shift invariant subspaces. Although the canonical choice of inner divisor seems natural, this choice does not behave as expected with respect to the lattice operations. The lattice of invariant subspaces is highly relevant to distance problems and the notion of hyperreflexivity of an operator algebra. We will say more about this later in this section. For this reason a better understanding of the structure of $\mathrm{Lat}(H^\infty_B)$ is needed. We do not, as yet, have at our disposal a useful way to describe the lattice operation in $\mathrm{Lat}(H^\infty_B)$. For illustrative purposes we examine what happens to the inner divisor when we take meets and joins of elements in $\mathrm{Lat}(H^\infty_B)$. Note that $\mathrm{Lat}(H^\infty)$ is a sublattice of $\mathrm{Lat}(H^\infty_B)$. Any good description of $\mathrm{Lat}(H^\infty_B)$ would have to take into account the fact that $\mathrm{Lat}(H^\infty)$ is the lattice of inner functions.

Let $\mathcal{M} = \phi_{\mathcal{M}}(W_{\mathcal{M}} \oplus BH^2), \mathcal{N} = \phi_{\mathcal{N}}(W_{\mathcal{N}} \oplus BH^2) \in \mathrm{Lat}(H^\infty_B)$, where $\phi_{\mathcal{M}}$ and

$\phi_{\mathcal{N}}$ are the inner divisors of $\mathcal{M}$ and $\mathcal{N}$ respectively. Let $\mathcal{X} = \mathcal{M} \cap \mathcal{N}$. We have,

$$\begin{aligned} B \operatorname{lcm}(\phi_{\mathcal{M}}, \phi_{\mathcal{N}}) H^2 &\subseteq \operatorname{lcm}(B\phi_{\mathcal{M}}, B\phi_{\mathcal{N}}) H^2 && (3.9) \\ &= (B\phi_{\mathcal{M}} H^2) \cap (B\phi_{\mathcal{N}} H^2) && (3.10) \\ &\subseteq \mathcal{M} \cap \mathcal{N} && (3.11) \\ &= \mathcal{X} && (3.12) \\ &\subseteq (\phi_{\mathcal{M}} H^2) \cap (\phi_{\mathcal{N}} H^2) && (3.13) \\ &= \operatorname{lcm}(\phi_{\mathcal{M}}, \phi_{\mathcal{N}}) H^2. && (3.14) \end{aligned}$$

Hence, $\phi_{\mathcal{X}}$ satisfies

$$\operatorname{lcm}(\phi_{\mathcal{M}}, \phi_{\mathcal{N}}) \leq \phi_{\mathcal{X}} \leq B \operatorname{lcm}(\phi_{\mathcal{M}}, \phi_{\mathcal{N}}). \quad (3.15)$$

These are the best general bounds we have. If we consider the case where $\phi_{\mathcal{M}} = \phi_{\mathcal{N}} = 1$, then we see that

$$\begin{aligned} \mathcal{X} &= \mathcal{M} \cap \mathcal{N} && (3.16) \\ &= (W_{\mathcal{M}} \cap W_{\mathcal{N}}) \oplus BH^2 && (3.17) \\ &= \gcd(W_{\mathcal{M}} \cap W_{\mathcal{N}}, B)(W_{\mathcal{X}} \oplus BH^2) && (3.18) \end{aligned}$$

If $W_{\mathcal{M}} \cap W_{\mathcal{N}} = \{0\}$, then the inner divisor $\phi_{\mathcal{X}} = B$. However, if $W_1 \cap W_2$ is non-trivial the situation can be different. Let $B = z^5$, let $\mathcal{M} = [1+z^2, z^3] \oplus z^5 H^2$ and let $\mathcal{N} = [1 - z^2, z^3] \oplus z^5 H^2$. It is straightforward to check that the inner divisor $\phi_{\mathcal{X}}$ of the intersection $\mathcal{X} = \mathcal{M} \cap \mathcal{N}$ is divisible by z^3. Since the functions $1+z^2$ and $1-z^2$

are outer we see that $\phi_{\mathcal{M}} = \phi_{\mathcal{N}} = 1$. Note that $\gcd(\phi_{\mathcal{M}}, \phi_{\mathcal{N}}) = 1 < \phi_{\mathcal{X}} < z^5 = B$.

If we consider the join of two subspaces $\mathcal{Y} = \mathcal{M} \vee \mathcal{N}$, then we have $\phi_{\mathcal{Y}} = \gcd(\phi_{\mathcal{M}}, \phi_{\mathcal{N}})$. The inequality $\gcd(\phi_{\mathcal{M}}, \phi_{\mathcal{N}}) \leq \phi_{\mathcal{Y}}$ follows from

$$\mathcal{M} \vee \mathcal{N} \subseteq (\phi_{\mathcal{M}} H^2) \vee (\phi_{\mathcal{N}} H^2) = \gcd(\phi_{\mathcal{M}}, \phi_{\mathcal{N}}). \tag{3.19}$$

Since $\phi_{\mathcal{Y}}|\mathcal{Y}$, we have $\phi_{\mathcal{Y}}|\mathcal{M}$ and $\phi_{\mathcal{Y}}|\mathcal{N}$. Hence, $\phi_{\mathcal{Y}}|\gcd(W, B)\phi_{\mathcal{M}} = \phi_{\mathcal{M}}$ and $\phi_{\mathcal{Y}}|\gcd(V, B)\phi_{\mathcal{N}} = \phi_{\mathcal{N}}$. Therefore, $\phi_{\mathcal{Y}} \leq \gcd(\phi_{\mathcal{M}}, \phi_{\mathcal{N}})$, which implies $\phi_{\mathcal{Y}} = \gcd(\phi_{\mathcal{M}}, \phi_{\mathcal{N}})$.

It is not difficult to extend Theorem 3.1.1 to the vector-valued setting. If $\mathcal{H}$ is a separable Hilbert space, then we denote by $H^2_{\mathcal{H}}$ the $\mathcal{H}$-valued Hardy space. The natural action of H^∞ on $H^2_{\mathcal{H}}$ is given by $(fh)(z) = f(z)h(z)$ and this makes $H^2_{\mathcal{H}}$ a module over H^∞. This action obviously restricts to H^∞_B and we say that a subspace $\mathcal{M}$ of $H^2_{\mathcal{H}}$ is invariant for H^∞_B if and only if $H^\infty_B \mathcal{M} \subseteq \mathcal{M}$. We denote by $H^\infty_{B(\mathcal{H})}$ the set of $B(\mathcal{H})$-valued bounded analytic functions. An element of $H^\infty_{B(\mathcal{H})}$ is called *rigid* if $\Phi(e^{i\theta})$ is a partial isometry a.e. on $\mathbb{T}$. A subspace $\mathcal{M}$ is invariant under H^∞ if and only if there exists a rigid function $\Phi \in H^\infty_{B(\mathcal{H})}$ such that $\mathcal{M} = \Phi H^2_{\mathcal{H}}$ [21]. The proof of the scalar case goes through with the obvious modifications to give the following result.

Theorem 3.1.4. *If $\mathcal{M}$ is a closed subspace of $H^2_{\mathcal{H}}$ which is invariant for H^∞_B, then there exists a rigid function $\Phi \in H^\infty_{B(H)}$ and a subspace $V \subseteq H^2_{\mathcal{H}} \ominus BH^2_{\mathcal{H}}$ such that $\mathcal{M} = \Phi(V \oplus BH^2_{\mathcal{H}})$.*

Proof. Let $\mathcal{M} \subseteq H^2_{\mathcal{H}}$ be an invariant subspace for H^∞_B. As in the proof of Theorem 3.1.1 we form the shift invariant subspace $[H^\infty \mathcal{M}] \subseteq H^2_{\mathcal{H}}$. By the invariant subspace theorem in [21] we can write $[H^\infty \mathcal{M}] = \Phi H^2_{\mathcal{H}}$ for a rigid function Φ. Now, $\mathcal{M} \supseteq [BH^\infty \mathcal{M}] = B[H^\infty \mathcal{M}] = B\Phi H^2_{\mathcal{H}}$ and so $B\Phi H^2_{\mathcal{H}} \subseteq \mathcal{M} \subseteq \Phi H^2_{\mathcal{H}}$. It follows that $\mathcal{M} = \mathcal{W} \oplus B\Phi H^2_{\mathcal{H}}$, where $\mathcal{W} \subseteq \Phi H^2_{\mathcal{H}} \ominus B\Phi H^2_{\mathcal{H}}$. If $w \in \mathcal{W}$, then $w = \Phi f$ for some $f \in H^2_{\mathcal{H}} \ominus BH^2_{\mathcal{H}}$. Choosing V to be the subspace of elements $f \in H^2_{\mathcal{H}} \ominus BH^2_{\mathcal{H}}$ such that $\Phi f \in \mathcal{W}$ completes the proof. □

This last result explains to some extent why we expect the matrix-valued result and scalar-valued result to be different. When studying the interpolation problem and the associated distance problem in the vector-valued setting we are interested in the vector-valued invariant subspaces for the algebra H^∞_B. In the vector-valued case, $H^2_{\mathcal{H}}$ is a direct sum of H^2 spaces and is essentially the only invariant subspace of H^∞ that needs to be considered. In the case of H^∞_B, the invariant subspaces for H^∞_B are of the form $V \oplus BH^2_{\mathcal{H}}$ with $V \subseteq H^2_{\mathcal{H}} \ominus BH^2_{\mathcal{H}}$ and these may fail to decompose as a direct sum of invariant subspaces contained in H^2. Therefore, one expects the scalar theory and vector-valued theory to be fundamentally different. A first indication of this fact is given by [15, Theorem 5.3] and Theorem 5.2.1 in Chapter 5.2 is an extension of this result.

In Chapter 3.2 we will prove a distance formula for H^∞_B. We will show that the distance of an element of L^∞ from the weak*-closed ideal in H^∞_B of functions that vanish at the n points $z_1, \ldots, z_n \in \mathbb{D}$ is related to the norm of the compression of M_f to the cyclic subspaces of H^∞_B. This result is an extension of Nehari's theorem and related to Arveson's distance formula for nest algebras [9].

To better understand these connections we need to look at the notions of *reflexivity* and *hyperreflexivity* for an operator algebra.

If $\mathcal{M}$ is a subspace of $\mathcal{H}$, then we denote by $P_{\mathcal{M}}$ the orthogonal projection from

$\mathcal{H}$ onto $\mathcal{M}$. Since $\mathcal{M}$ is invariant for an operator A if and only if $P_{\mathcal{M}}AP_{\mathcal{M}} = AP_{\mathcal{M}}$ we see that $\mathrm{Lat}(\mathcal{A})$ can be viewed as the set of projections $P \in B(\mathcal{H})$ such that $PAP = AP$ for all $A \in \mathcal{A}$. If $\tilde{\mathcal{A}}$ is the closure in the WOT of the algebra generated by $\mathcal{A}$ and I, then $\mathrm{Lat}(\tilde{\mathcal{A}}) = \mathrm{Lat}(\mathcal{A})$. Therefore, we assume that our algebras are unital and closed in the WOT.

The ideas above can be dualized. Given a lattice $\mathcal{L}$ of subspaces, equivalently projections, the algebra $\mathrm{Alg}(\mathcal{L})$ is defined as the set of operators on $\mathcal{H}$ which leave every element of $\mathcal{L}$ invariant. It is straightforward that $\mathcal{A} \subseteq \mathrm{Alg}(\mathrm{Lat}(\mathcal{A}))$. An algebra for which this last inclusion is an equality is called *reflexive*. In the case where $\mathcal{L}$ is a chain the algebra $\mathrm{Alg}(\mathcal{L})$ is called a nest algebra and nest algebras are reflexive. Nest algebras have a property that is stronger than reflexivity, nest algebras are *hyperreflexive* [9]. An algebra $\mathcal{A}$ is reflexive if and only if $\|(I-P)TP\| = 0$ for all $P \in \mathrm{Lat}(\mathcal{A})$ implies $T \in \mathcal{A}$.

If $P \in \mathrm{Lat}(\mathcal{A})$, $A \in \mathcal{A}$ and $T \in B(\mathcal{H})$, then

$$\|T + A\| \geq \|(I-P)(T+A)P\| = \|(I-P)TP\|. \tag{3.20}$$

Hence,

$$\|T + \mathcal{A}\| = \inf_{A \in \mathcal{A}} \|T + A\| \geq \sup_{P \in \mathrm{Lat}(\mathcal{A})} \|(I-P)TP\|. \tag{3.21}$$

If the reverse inequality in (3.21) holds, even up to a constant, then $\mathcal{A}$ is called hyperreflexive and this property implies that $\mathcal{A}$ is reflexive.

If $\mathcal{L}$ is the lattice in $B(H^2)$ of subspaces of the form ϕH^2 where ϕ is an inner function, then the $\mathrm{Alg}(\mathrm{Lat}(\mathcal{L})) = H^\infty$ and so H^∞ is reflexive. This latter fact holds true for the algebras H^∞_B when viewed as a subalgebra of $B(H^2)$ and for the algebra H^∞_Γ when viewed as operators on H^2_Γ.

Proposition 3.1.5. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space. If $\mathcal{A}$ denotes the multiplier algebra of $\mathcal{H}$, then $\mathcal{A}$ is a reflexive operator algebra in $B(\mathcal{H})$.*

Proof. Suppose that $T \in B(\mathcal{H})$ and $T(M) \subseteq M$ for all $M \in \mathcal{L}(\mathcal{A})$. The subspace spanned by the kernel function k_x is invariant for $\mathcal{A}^*$ and so $T^*k_x \in \operatorname{span}\{k_x\}$. There exists a constant $\phi(x) \in \mathbb{C}$ such that $T^*k_x = \phi(x)k_x$ and consequently $T = M_{\overline{\phi}}$. □

In order to apply this last result we must know that H^∞_Γ is the multiplier algebra of H^2_Γ and that H^∞_B is the multiplier algebra of H^2_B. This is established in Proposition 4.1.3.

When H^∞ is viewed as a subalgebra of $B(H^2)$ it is well known that H^∞ is equal to its own commutant. In fact, the commutant lifting approach to interpolation depends on the fact that $H^\infty = \{S\}'$. The next few results show what we know about the commutant of the algebras H^∞_B and H^∞_Γ.

If $\gamma : \mathbb{D} \to \mathbb{D}$ is an automorphism, then the *composition operator* $C_\gamma : H^2 \to H^2$ is defined by $C_\gamma(f) = f \circ \gamma$. It is a consequence of Littlewood's theorem that this operator is bounded and an easy calculation,

$$\left\langle f, C_\gamma^* k_z \right\rangle = \langle C_\gamma f, k_z \rangle = f(\gamma(z)) = \left\langle f, k_{\gamma(z)} \right\rangle, \tag{3.22}$$

shows that $C_\gamma^* k_z = k_{\gamma(z)}$.

Given a Fuchsian group Γ, let us denote by $\mathrm{Alg}(S, \Gamma)$ the smallest subalgebra of $B(H^2)$ that contains the shift S and the group of composition operators $\{C_\gamma : \gamma \in \Gamma\}$.

If $h \in H^2$, $f \in H^\infty$ and $\gamma \in \Gamma$, then

$$\begin{aligned} M_f C_\gamma(h) &= f(h \circ \gamma) && (3.23)\\ &= (f \circ \gamma^{-1}\gamma)(h \circ \gamma) && (3.24)\\ &= (f \circ \gamma^{-1}h) \circ \gamma && (3.25)\\ &= C_\gamma M_{f\circ\gamma^{-1}}(h). && (3.26) \end{aligned}$$

Hence, $M_{f\circ\gamma^{-1}} = C_\gamma^{-1} M_f C_\gamma$. This shows that the action of Γ on H^∞ is implemented by a similarity. The fixed point space for this action is the algebra H^∞_Γ which is the set of $f \in H^\infty$ such that $M_f C_\gamma = C_\gamma M_f$ for all $\gamma \in \Gamma$. The closure of the algebra $\mathrm{Alg}(S, \Gamma)$ in the WOT would seem a natural way to define a cross product of H^∞ by the group Γ. We have not explored this construction, nor does it seem to appear in the literature, but we feel it would be of interest to do so.

Proposition 3.1.6. *The commutant of* $\mathrm{Alg}(S, \Gamma)$ *in* $B(H^2)$ *is* H^∞_Γ.

Proof. If $T \in \mathrm{Alg}(S, \Gamma)'$, then $TS = ST$ which forces $T = M_f \in H^\infty$. Since $M_f C_\gamma = C_\gamma M_f$, for all $\gamma \in \Gamma$, we get $f \in H^\infty_\Gamma$. □

Proposition 3.1.7. *The invariant subspaces of* $\mathrm{Alg}(S, \Gamma)$ *are of the form* ϕH^2*, where* ϕ *is character automorphic.*

Proof. If $\mathcal{M}$ is invariant for $\mathrm{Alg}(S, \Gamma)$, then $\mathcal{M}$ is shift invariant. It follows that $\mathcal{M} = \phi H^2$ for some inner function ϕ. Since $\mathcal{M}$ is invariant under C_γ we get that $(\phi \circ \gamma)H^2 = \phi H^2$. It follows that $\phi \circ \gamma = \chi(\gamma)\phi$ where $\chi(\gamma) \in \mathbb{T}$. If $\gamma_1, \gamma_2 \in \Gamma$, then

$$\chi(\gamma_1\gamma_2)\phi = \phi \circ (\gamma_1\gamma_2) \tag{3.27}$$

$$= (\phi \circ \gamma_1) \circ \gamma_2 \tag{3.28}$$

$$= (\chi(\gamma_1)\phi) \circ \gamma_2 \tag{3.29}$$

$$= \chi(\gamma_1)\chi(\gamma_2)\phi. \tag{3.30}$$

Hence, $\chi \in \hat{\Gamma}$. □

Proposition 3.1.8 implies that H_B^∞ is equal to its own commutant when viewed as the multiplier algebra of H_B^2. The result also applies to H_Γ^∞ viewed as a subalgebra of $B(H_\Gamma^2)$.

Proposition 3.1.8. *Let $\mathcal{M}$ be a subspace of H^2 such that $1 \in \mathcal{M}$ and let $\mathcal{A}$ be the multiplier algebra of this subspace. If $[\mathcal{A}] = \mathcal{M}$, then $\mathcal{A}' = \mathcal{A}$, when $\mathcal{A}$ is represented as multiplication operators in $B(\mathcal{M})$.*

Proof. Let $T \in B(\mathcal{M})$ such that $T \in \mathcal{A}'$ and set $T(1) = h$. Consider the action of T^* on the kernel function $k_x \in \mathcal{M}$. Let $g \in \mathcal{A}$ and compute

$$\langle T^* k_x, g\rangle = \langle k_x, TM_g 1\rangle$$

$$= \langle k_x, M_g T(1)\rangle \tag{3.31}$$

$$= \langle k_x, gh\rangle \tag{3.32}$$

$$= \overline{g(x)h(x)} \tag{3.33}$$

$$= \left\langle \overline{h(x)} k_x, g \right\rangle \tag{3.34}$$

and so $T^* k_x = \overline{h(x)} k_x$. This forces h to be in the multiplier algebra and $T = M_h \in \mathcal{A}$. □

Besides the representation on H^2_Γ, Forelli [20] showed that there is a natural representation of H^∞_Γ on the space $H^2_\Gamma + N$. The subspace $H^2_\Gamma + N$ is invariant for H^∞_Γ. To see this pick $h \in H^2_\Gamma + N$ and choose $k \in H^2$ such that $E(k) = h$. If $f \in \overline{H^2_{0,\Gamma}}$ and $g \in H^\infty_\Gamma$, then

$$\int f\overline{gh} = \int f\overline{g}\overline{\Phi(k)} = \int f\overline{gk} = 0, \tag{3.35}$$

Hence, $H^\infty_\Gamma(H^2_\Gamma + N) \subseteq H^2_\Gamma + N$. This shows that N is semi-invariant for H^∞_Γ and so the compression of H^∞_Γ to N is a homomorphism of H^∞_Γ. Since H^∞_Γ is infinite-dimensional and N is finite-dimensional we see that this homomorphism has a non-trivial kernel. Hence, there exists a function $f \in H^\infty_\Gamma$ such that $M_f(N) \perp N$. Let $\mathcal{N}$ be the closure in H^2 of the functions f in H^2 such that $fN \subset H^2$. The subspace $\mathcal{N}$ is invariant for $\mathrm{Alg}(S, \Gamma)$ which tells us that $\phi H^2 = \mathcal{N}$ for a character automorphic inner function ϕ, by Proposition 3.1.7. This means that $\phi N \subseteq H^\infty$ with ϕ character automorphic.

We point out that there is a more elementary way to obtain this last fact. Let $\mathcal{M}$ denote the smallest shift invariant subspace of L^2 that contains $H^2_\Gamma + N$. Clearly $\mathcal{M} = [H^\infty(H^2_\Gamma + N)]$. By the Helson-Lowdenslager theorem the subspace is either of the form $\chi_S L^2$ or of the form ϕH^2 where ϕ is unimodular. Suppose that $\mathcal{M} = \chi_S L^2$ and note that $\chi_S = 1$, since $1 \in \mathcal{M}$. If $\mathcal{M} = L^2$, then $\Phi(\mathcal{M}) = L^2_\Gamma$. A typical element of $\mathcal{M}$ can be approximated by sums of elements of the form fg where $f \in H^\infty$ and $g \in H^2_\Gamma + N$. Note that $\Phi(fg) = \Phi(f)g$, which shows

$$L^2_\Gamma = \Phi(\mathcal{M}) \subseteq [\{\Phi(fg) : f \in H^\infty, g \in H^2_\Gamma + N\}] \tag{3.36}$$

$$= [\{\Phi(f)g : f \in H^\infty, g \in H^2_\Gamma + N\}] \tag{3.37}$$

$$= [\{fg : f \in H^\infty_\Gamma + N, g \in H^2_\Gamma + N\}] \tag{3.38}$$

$$\subseteq H^2_\Gamma + N + N.N, \tag{3.39}$$

which is impossible since N is finite dimensional and $\overline{H^\infty_{0,\Gamma}}$ is not. Hence, $\mathcal{M} = \psi H^2$ for a unimodular character automorphic function ψ. Since $1 \in H^2_\Gamma + N \subseteq \psi H^2$ we get $1 = \psi\phi$ for an inner function ϕ and so $\overline{\psi} = \phi$ is inner. Hence, $\phi(H^2_\Gamma + N) \subseteq H^2$. This ability to multiply N into H^∞ with a character automorphic inner function is critical to our proof of Theorem 3.2.10 and so we record this fact.

Proposition 3.1.9 (Forelli)**.** *There exists a character automorphic inner function ϕ such that $\phi N \subseteq H^\infty$.*

3.2 Distance formulae

Let $\mathcal{A}$ be a unital, weak*-closed subalgebra of H^∞. Let $z_1, \ldots, z_n \in \mathbb{D}$, $w_1, \ldots, w_n \in \mathbb{C}$ and assume that $\mathcal{A}$ contains a function f such that $f(z_j) = w_j$. We saw in Chapter 1.1 how the distance of an element in the algebra $\mathcal{A}$ from the weak*-closed ideal $\mathcal{I}$ of functions in $\mathcal{A}$ which vanish at the n points $z_1, \ldots, z_n$ is related to the Nevanlinna-Pick problem.

The following lemma, the proof of which follows from a standard weak*-compactness argument, shows that when dealing with weak*-closed subalgebras of H^∞, the existence of a solution to the interpolation problem is equivalent to the existence of an interpolating function and the condition $\|f + \mathcal{I}\| \leq 1$.

Lemma 3.2.1. *Let $f \in \mathcal{A}$ with $f(z_j) = w_j$. A solution to the interpolation problem exists if and only if $\|f + \mathcal{I}\| \leq 1$.*

In this section we will prove two distance formulae. The first formula relates the distance of an element of L^∞ to the ideal $\mathcal{I}$ in an algebra of the form $\mathcal{A} := \bigcap_{j \in J} H^\infty_{B_j}$ where B_j is an inner function and J is a set. One special case of this is the algebra H^∞_Γ and so we have our first distance formula for the norm on $H^\infty_\Gamma/\mathcal{I}$. Our second result uses the additional structure present in the case of H^∞_Γ to derive a formula for the distance of an element in L^∞_Γ from the ideal $\mathcal{I}$ in H^∞_Γ. This improved result will be used in Chapter 4.1 to generalize Abrahamse's theorem.

We identify L^∞ as the dual of L^1 and refer to L^1 as the predual of L^∞. Proposition 2.3.1 gives the corresponding results for the L^p_Γ spaces.

We will now introduce a property which we call *predual factorization*. If we examine closely the essential aspects of Sarason's generalized interpolation [37] and Abrahamse's theorem [1] we see that this notion is natural. It allows us to capture the essential aspects of Sarason's idea. We say a subspace $\mathcal{X} \subseteq H^\infty$ has *predual factorization* if and only if there exists a (not necessarily closed) subspace $\mathcal{S} \subseteq L^1$ with the following two properties.

1. The closure of $\mathcal{S}$ in the L^1 norm is $\mathcal{X}_\perp$
2. For each $f \in \mathcal{S}$, there exists an inner function ϕ such that $\phi f \in H^1$.

A simple consequence of Riesz factorization is that any function $f \in \mathcal{S}$ can be written as $f = \psi u^2$ where ψ is unimodular, u is outer and $|f|^{1/2} = |u|$.

Proposition 3.2.2. *Suppose that $\{\mathcal{X}_j : j \in J\}$ is a set of weak*-closed subspaces* *of H^∞. If for each $j \in J$, $\mathcal{X}_j$ has predual factorization, then $\mathcal{X} := \bigcap_{j\in J} \mathcal{X}_j$ has predual factorization.*

Proof. We note that

$$\mathcal{X}_\perp = \left(\bigcap_{j\in J} \mathcal{X}_j\right)_\perp = \left[\bigcup_{j\in J} \{(\mathcal{X}_j)_\perp \,:\, j \in J\}\right]_1 . \tag{3.40}$$

Set $\mathcal{S} = \operatorname{span}\{(\mathcal{X}_j)_\perp \,:\, j \in J\}$. Given $f_j \in \mathcal{X}_j$ there exists an inner function ϕ_j such that $\phi_j f_j \in H^1$. If $f = \sum_{i=1}^m c_i f_{j_i} \in \mathcal{S}$, then $\phi f \in H^1$ where $\phi = \phi_{j_1} \cdots \phi_{j_m}$. Hence, $\mathcal{S}$ has predual factorization. $\square$

Proposition 3.2.3. *If $\mathcal{X}$ is a subspace of L^∞ such that $BH^\infty \subseteq \mathcal{X}$, then $\mathcal{X}$ has predual factorization.*

Proof. We have $\mathcal{X}_\perp \subseteq (BH^\infty)_\perp = \overline{B}H^1_0$. The inner function B multiplies $\mathcal{X}_\perp$ into H^1. $\square$

Corollary 3.2.4. *If $\{B_j \,:\, j \in J\}$ is a set of inner functions, and $\mathcal{X}_j \supseteq B_j H^\infty$, then $\mathcal{X} = \bigcap_{j\in J} \mathcal{X}_j$ has predual factorization.*

Corollary 3.2.5. *If $\{B_j \,:\, j \in J\}$ is a set of inner functions, then $\mathcal{A} = \bigcap_{j\in J} H^\infty_{B_j}$ has predual factorization.*

Recall that a function $u \in H^2$ is called outer if $[H^\infty u] = H^2$. Given an outer function $u \in H^2$ we define $\mathcal{M}_u = [\mathcal{A}u]$, $\mathcal{K}_u$ to be the span of the kernel functions for $\mathcal{M}_u$ at the points $z_1, \ldots, z_n$ and $\mathcal{N}_u = \mathcal{M}_u \ominus \mathcal{K}_u = \{f \in \mathcal{M}_u \,:\, f(z_j) = 0, j = 1, \ldots, n\}$. Given a subspace $\mathcal{M} \subseteq L^2$ we denote by $P_\mathcal{M}$ the orthogonal projection of L^2 onto $\mathcal{M}$.

Lemma 3.2.6. *let $z_1, \dots, z_n$ be n distinct points in $\mathbb{D}$ and suppose $\mathcal{A}$ has predual factorization. If $\mathcal{I}$ is the ideal of functions in $\mathcal{A}$ such that $f(z_j) = 0$ for $j = 1, \dots, n$, then $\mathcal{I}$ has predual factorization.*

Proof. Let $\mathcal{I}_\perp$ be the preannihilator of $\mathcal{I}$ in L^1. Since $\mathcal{A}$ has predual factorization there exists $\mathcal{S} \subseteq \mathcal{A}_\perp$ such that

1. The closure of $\mathcal{S}$ in the L^1 norm is $\mathcal{A}_\perp$
2. For each $f \in \mathcal{S}$, there exists an inner function ϕ such that $\phi f \in H^1$.

Note that $\mathcal{I}_\perp = \mathcal{A}_\perp + \mathrm{span}\{\overline{k_{z_j}} : 1 \leq j \leq n\}$, where k_z is the Szegö kernel at the point z. If E is the Blaschke product for the points $z_1, \dots, z_n$, then $E\overline{k_{z_j}} \in H^\infty$ for $j = 1, \dots, n$. The space $\tilde{\mathcal{S}} = \mathcal{S} + \mathrm{span}\{\overline{k_{z_j}} : 1 \leq j \leq n\}$ is dense in $\mathcal{I}_\perp$. Given $h + v \in \tilde{\mathcal{S}}$, with $h \in \mathcal{A}_\perp$ and $v \in \mathrm{span}\{\overline{k_{z_j}} : 1 \leq j \leq n\}$, there exists an inner function ϕ such that $\phi h \in H^1$ and so $E\phi(h + v) \in H^1$ and $E\phi$ is inner. Hence $\mathcal{I}_\perp$ has predual factorization. □

Lemma 3.2.7. *Let $\mathcal{I}$, $\mathcal{A}$ be as in Lemma 3.2.6. If u is an outer function, then $[\mathcal{I}u] = \mathcal{N}_u$.*

Proof. Since every function in $\mathcal{I}$ vanishes at $z_1, \dots, z_n$, $[\mathcal{I}u] \subseteq \mathcal{N}_u$. On the other hand, given $f \in \mathcal{N}_u$ we know that there exists $f_m \in \mathcal{A}$ such that $\|f_m u - f\|_2 \to 0$ and since u does not vanish at any point of the disk we see that $f_m(z_j) \to 0$ for $j = 1, \dots, n$. By a construction similar to the one in Lemma 4.1.5 we see that there exist functions $e_j \in \mathcal{A}$ such that $e_j(z_i) = \delta_{i,j}$. Setting $g_m = f_m - \sum_{i=1}^n f_m(z_i)e_i$ we see that $g_m u$ converges to f in H^2 and $g_m \in \mathcal{I}$. Hence, $\mathcal{N}_u \subseteq [\mathcal{I}u]$ and our proof is complete. □

We will now prove our distance formula.

Theorem 3.2.8. *Let $z_1, \ldots, z_n$ be n distinct points in $\mathbb{D}$, let $\mathcal{A}$ be a unital, weak*-closed subalgebra of H^∞ with predual factorization and let $\mathcal{I}$ be the ideal of functions in $\mathcal{A}$ such that $f(z_j) = 0$ for $j = 1, \ldots, n$. If $f \in L^\infty$, then*

$$\|f + \mathcal{I}\| = \sup_u \|(I - P_{\mathcal{N}_u}) M_f P_{\mathcal{M}_u}\|, \tag{3.41}$$

where the supremum is taken over all outer functions $u \in H^2$.

Proof. We have,

$$\|f + \mathcal{I}\| = \sup \left\{ \left| \int fg \right| \,:\, g \in \mathcal{I}_\perp, \|g\|_1 \le 1 \right\} \tag{3.42}$$

$$= \sup \left\{ \left| \int fg \right| \,:\, g \in \tilde{\mathcal{S}}, \|g\|_1 \le 1 \right\}, \tag{3.43}$$

where $\tilde{S}$ is a dense subspace of $\mathcal{I}_\perp$ with the property that each function in $\mathcal{S}$ can be multiplied into H^1 by an inner function. Let $g \in \tilde{\mathcal{S}}$, and let ϕ be inner with the property that $\phi g \in H^1$ and factor ϕg as $g_1 u$ where $g_1, u \in H^2$, u is outer, and $\|u\|_2 = \|g_1\|_2 = \|g\|_1^{1/2}$. It follows that $g = g_2 u$ where $g_2 \in L^2$ and u is outer with $\|u\|_2 = \|g_2\|_2 = \|g\|_1^{1/2}$.

Since $g \in \mathcal{I}_\perp$, for all $h \in \mathcal{I}$ we get

$$0 = \int gh = \int g_2 uh = \langle hu, \overline{g_2} \rangle. \tag{3.44}$$

This shows $\overline{g_2} \perp [\mathcal{I}u] = \mathcal{N}_u$. Hence,

$$\left|\int fg\right| = |\langle fu, \overline{g_2}\rangle| \tag{3.45}$$

$$= |\langle fP_{\mathcal{M}_u}u, (I - P_{\mathcal{N}_u})\overline{g_2}\rangle| \tag{3.46}$$

$$\leq \|(I - P_{\mathcal{N}_u})M_f P_{\mathcal{M}_u}\|. \tag{3.47}$$

Recall from our statements about reflexivity that the reverse inequality is always true. Our proof is complete. □

We point out that the proof of Theorem 3.2.8 holds in the case $n = 0$ to give the distance of an element in L^∞ from the algebra $\mathcal{A} = \bigcap_{j\in J} H^\infty_{B_j}$. This result can be interpreted as a Nehari theorem for the algebra $\mathcal{A}$.

Theorem 3.2.9. *If $f \in L^\infty$, then $\|f + \mathcal{A}\| = \sup_u \|(I - P_{\mathcal{M}_u})M_f P_{\mathcal{M}_u}\|$.*

A more refined version of this formula can be obtained for the distance of $f \in L^\infty_\Gamma$ from the ideal $\mathcal{I} \subseteq H^\infty_\Gamma$. The preannihilator of H^∞_Γ in L^1_Γ is $H^1_{0,\Gamma} + N$. Let $g \in H^1_{0,\Gamma} + N$. We have seen in Proposition 3.1.9 that there is an inner function ϕ such that $\phi N \subseteq H^1$ and so $\phi g \in H^1$. Hence, $|g| = |u|^2$ for some outer function u. The function g is in L^∞_Γ and so $u \in H^2_\chi$ for some $\chi \in \hat{\Gamma}$ by Proposition 2.2.1.

Theorem 3.2.10. *Let $f \in L^\infty_\Gamma$. The distance of f from $\mathcal{I}$ is given by*

$$\|f + \mathcal{I}\| = \sup \|(I - P_{\mathcal{N}_u})M_f P_{\mathcal{M}_u}\|, \tag{3.48}$$

where the supremum is over all characters $\chi \in \hat{\Gamma}$ and all outer functions u in H^2_χ.

Here I denotes the identity in $B(L^2)$ and the orthogonal projections are in $B(L^2)$.

Proof. Let $k_{z_j}^{\Gamma}$ be the kernel function at the point z_j for the space H^2_{Γ}. By duality

$$\|f+\mathcal{I}\| = \sup \left| \int fg \right| \tag{3.49}$$

where $g \in H^1_{\Gamma,0} + N + \text{span}\{\overline{k_{z_1}^{\Gamma}}, \dots, \overline{k_{z_n}^{\Gamma}}\}$, $\|g\|_1 \leq 1$. If we let B denote the Blaschke whose zero set is the union of the orbits $\Gamma(z_1), \dots, \Gamma(z_n)$, then $B\overline{k_{z_j}^{\Gamma}} \in H^2$. It follows that $\phi Bg \in H^1$ and so $|g| = |u|^2$, where $u \in H^2_{\chi}$ is outer and $\chi \in \hat{\Gamma}$. Rewriting the expression in (3.49) we get

$$\|f+\mathcal{I}\| = \sup \left| \int fu\overline{v} \right| \tag{3.50}$$

where u is an outer function in H^2_{χ} and $\overline{v}u = g$. Since $g \in \mathcal{I}_{\perp}$ we see that $\int gh = 0$ for all $h \in \mathcal{I}$. Therefore, $\langle hu, v\rangle = 0$ and so $v \in L^2 \ominus [\mathcal{I}u] = L^2 \ominus \mathcal{N}_u$. Hence,

$$\left| \int fu\overline{v} \right| = |\langle fu, v\rangle| \leq \|(I - P_{\mathcal{N}_u})M_f P_{\mathcal{M}_u}\|. \tag{3.51}$$

The reverse inequality is straightforward since $\mathcal{M}_u$ is an invariant subspace for the algebra H^{∞}_{Γ}. $\square$

Chapter 4

Interpolation Results

4.1 Nevanlinna-Pick interpolation for H_B^∞

In this section we will prove the interpolation theorem for H_B^∞. This result is a consequence of the following, more general, theorem.

Theorem 4.1.1. *Let $\mathcal{A}$ be a weak*-closed, unital subalgebra of H^∞ which has predual factorization. Let $z_1, \ldots, z_n \in \mathbb{D}$ and $w_1, \ldots, w_n \in \mathbb{C}$. There exists a function $f \in \mathcal{A}$ such that $f(z_j) = w_j$, $j = 1, \ldots, n$ with $\|f\|_\infty \leq 1$ if and only if for all outer functions $u \in H^2$, $[(1 - w_i\overline{w_j})K^u(z_i, z_j)]_{i,j=1}^n \geq 0$.*

Here K^u denotes the kernel function of the space $[\mathcal{A}u]$. The proof of this result will follow from the distance formula obtained in Theorem 3.2.8.

We first need to establish the fact that the multiplier algebra of $\mathcal{H} = \bigcap_{j\in J} H^2_{B_j}$ is $\mathcal{A} = \bigcap_{j\in J} H^\infty_{B_j}$ and that the supremum norm agrees with the multiplier norm.

Proposition 4.1.2. *Let $\mathcal{A}$ be a unital, weak*-closed subalgebra of H^∞ and let $u \in H^2$ be an outer function. If $\mathcal{M} := [\mathcal{A}u]$, then $\mathcal{A} \subseteq \mathrm{mult}(\mathcal{M})$.*

Proof. It is straightforward that $\mathcal{A}(\mathcal{M}) \subseteq \mathcal{M}$. Since u does not vanish on the disk we see that none of the kernel functions in $\mathcal{M}$ are the zero function. If M_f denotes the multiplication operator on $\mathcal{M}$ induced by f, then $\|f\|_{\text{mult}} = \|M_f\|_{B(\mathcal{M})} \geq \|f\|_\infty$. On the other hand if $h \in \mathcal{M} \subseteq L^2$, then

$$\|M_f h\|^2 = \int |fh|^2 \leq \|f\|_\infty^2 \|h\|^2, \tag{4.1}$$

which proves $\|f\|_{\text{mult}} \leq \|f\|_\infty$. □

Proposition 4.1.3. *Let* $\{B_j : j \in J\}$ *be a set of inner functions. The multiplier algebra* $\text{mult}(\bigcap_{j\in J} H^2_{B_j}) = \bigcap_{j\in J} H^\infty_{B_j}$.

Proof. Let us denote $\mathcal{M} := \bigcap_{j\in J} H^2_{B_j}$. Let $f \in \text{mult}(\mathcal{M})$. Since $1 \in \mathcal{M}$ none of the kernel functions in $\mathcal{M}$ can be zero. This shows that any $f \in \text{mult}(\mathcal{M})$ must be bounded. If $f \in \text{mult}(\mathcal{M})$, then $f \in \mathcal{M}$, since $1 \in \mathcal{M}$. Hence, $f = \lambda_j + B_j k_j$ for $j \in J$, $k_j \in H^2$. Since f is bounded so is k_j and we have shown that $f \in \bigcap_{j\in J} H^\infty_{B_j}$. On the other hand any function $f \in \bigcap_{j\in J} H^\infty_{B_j}$ multiplies $\mathcal{M}$ into itself. It remains to be seen that $\|f\|_{\text{mult}} \leq \|f\|_\infty$. This follows from

$$\|M_f h\|^2 = \int |fh|^2 \leq \|f\|_\infty^2 \|h\|^2, \tag{4.2}$$

where $h \in \mathcal{M} \subseteq L^2$. □

If $f \in \mathcal{A}$, then M_f leaves $\mathcal{M}_u$ invariant and

$$\|f + \mathcal{I}\| = \sup_u \|(I - P_{\mathcal{N}_u}) M_f P_{\mathcal{M}_u}\| \tag{4.3}$$

$$= \sup_u \|(I - P_{\mathcal{M}_u} + P_{\mathcal{K}_u}) M_f P_{\mathcal{M}_u}\| \tag{4.4}$$

$$= \sup_u \|P_{\mathcal{K}_u} M_f P_{\mathcal{M}_u}\| \tag{4.5}$$

$$= \sup_u \left\|M_f^* P_{\mathcal{K}_u}\right\|. \tag{4.6}$$

If k_z^u denotes the kernel function for $\mathcal{M}_u$ at z, then a spanning set for $\mathcal{K}_u$ is given by $\{k_{z_1}^u, \ldots, k_{z_n}^u\}$. Standard results about multiplier algebras of reproducing kernel Hilbert spaces tell us that the norm of $M_f^* P_{\mathcal{K}_u}$ is at most 1 if and only if $[(1 - w_i\overline{w_j})K^u(z_i, z_j)]_{i,j=1}^n \geq 0$. Combining this fact with equation (4.3)–(4.6) proves the interpolation theorem, once we know that there exists at least one interpolating function $f \in \mathcal{A}$ such that $f(z_j) = w_j$.

The purpose of the next two lemmas is to show that if the matrix

$$[(1 - w_i\overline{w_j})K^u(z_i, z_j)]_{i,j=1}^n \tag{4.7}$$

is positive for just one outer function $u \in H^2$, then there exists an interpolating function for the algebra $\mathcal{A}$.

Lemma 4.1.4. *Let $\mathcal{H}$ be a Hilbert space, let $v_1, \ldots, v_n \in \mathcal{H}$ and let $W_1, \ldots, W_n$, $W_{n+1} \in M_N$. Suppose that $v_{n+1} \in \mathcal{H}$ and $v_{n+1} = \sum_{i=1}^n \alpha_i v_i$. If the matrix*

$$Q = \left[(I - W_i W_j^*) \langle v_j, v_i\rangle\right]_{i,j=1}^{n+1} \tag{4.8}$$

is positive and $1 \leq i \leq n$, then either $\alpha_i = 0$ or $W_i = W_{n+1}$.

Proof. Let $W_k = \left[w_{i,j}^{(k)}\right]_{i,j=1}^N$ and consider the matrix, Q_k that we get by compressing to the (k, k) entry of each block in Q. The (i, j)-th entry of Q_k is

$\left(1-\sum_{l=1}^{n+1} w_{k,l}^{(i)}\overline{w_{k,l}^{(j)}}\right)\langle v_j, v_i\rangle$. Let $\lambda_1,\dots,\lambda_{n+1}\in\mathbb{C}$ and note that

$$\sum_{i,j=1}^{n+1}\left(1-\sum_{l=1}^{n+1} w_{k,l}^{(i)}\overline{w_{k,l}^{(j)}}\right)\langle v_j, v_i\rangle\,\overline{\lambda_i}\lambda_j\geq 0. \tag{4.9}$$

By setting $\lambda_j=\alpha_j$ for $j=1,\dots,n$ and $\lambda_{n+1}=-1$ we get that

$$\sum_{i,j=1}^{n}\left(1-\sum_{l=1}^{n+1} w_{k,l}^{(i)}\overline{w_{k,l}^{(j)}}\right)\langle v_j, v_i\rangle\,\overline{\alpha_i}\alpha_j-\sum_{i=1}^{n}\left(1-\sum_{l=1}^{n+1} w_{k,l}^{(i)}\overline{w_{k,l}^{(n+1)}}\right)\langle v_{n+1}, v_i\rangle\,\overline{\alpha_i} \tag{4.10}$$

$$-\sum_{j=1}^{n}\left(1-\sum_{l=1}^{n+1} w_{k,l}^{(n+1)}\overline{w_{k,l}^{(j)}}\right)\langle v_j, v_{n+1}\rangle\,\alpha_j+\left(1-\sum_{l=1}^{n+1}\left|w_{k,l}^{(n+1)}\right|^2\right)\|v_{n+1}\|^2\geq 0 \tag{4.11}$$

This simplifies to

$$\left\|v_{n+1}-\sum_{i=1}^{n}\alpha_i v_i\right\|^2-\sum_{l=1}^{n+1}\left\|\sum_{i=1}^{n}(w_{k,l}^{(n+1)}-w_{k,l}^{(i)})\alpha_i v_i\right\|^2\geq 0 \tag{4.12}$$

which gives $\sum_{i=1}^{n}(w_{k,l}^{(n+1)}-w_{k,l}^{(i)})\alpha_i v_i=0$ for $1\leq k,l\leq n$. If $\alpha_i\neq 0$, then by the linear independence of $v_1,\dots,v_n$ we get $w_{k,l}^{(n+1)}=w_{k,l}^{(i)}$ and so $W_{n+1}=W_i$. $\square$

Lemma 4.1.5. *Let $\mathcal{A}$ be a unital, weak*-closed subalgebra of H^∞. Let u be an outer function and let $\mathcal{M}=[\mathcal{A}u]$. Let K be the kernel function of $\mathcal{M}$, $z_1,\dots,z_n$ be n points in the disk and $W_1,\dots,W_n\in M_k$. If $\left[(1-W_iW_j^*)K(z_i,z_j)\right]_{i,j=1}^n\geq 0$, then there exists $F\in M_k(\mathcal{A})$ such that $F(z_j)=W_j$.*

Proof. We may assume after reordering the points that $\{k_{z_1},\dots.k_{z_m}\}$ is basis of $\operatorname{span}\{k_{z_j}\,:\,1\leq j\leq n\}$, with $m\leq n$. There exists $f\in\mathcal{M}$ such that $u(z_j)f(z_j)$

are distinct for for $1 \leq j \leq m$. If this is not the case, then $\overline{u(z_j)}k_{z_j} - \overline{u(z_i)}k_{z_i} = 0$ is a non-trivial linear combination of k_{z_i} and k_{z_j}, since $u(z_i), u(z_j) \neq 0$. From the definition of $\mathcal{M}$ we conclude that there exists $g \in \mathcal{A}$ such that $\|ug - f\|$ is small enough that $g(z_j)$ are distinct for $1 \leq j \leq m$. By setting

$$e_j = \prod_{r=1, r\neq j}^{m} \frac{g - g(z_r)}{g(z_j) - g(z_r)} \in \mathcal{A}, \tag{4.13}$$

we see that $e_i(z_j) = \delta_{i,j}$, for $1 \leq i, j \leq m$. Let $h = \sum_{i=1}^m w_i e_i$ and note that for $1 \leq j \leq m$, $h(z_j) = w_j$. To complete the proof we need to show that $h(z_j) = w_j$ for $j > m$.

Let $j > m$ and suppose that $k_{z_j} = \sum_{l=1}^m \alpha_l k_{z_l}$. We have seen that the matrix positivity condition implies that either $w_j = w_l$ or $\alpha_l = 0$ for $1 \leq l \leq m$. Hence,

$$h(z_j) = \sum_{i=1}^m w_i e_i(z_j) \tag{4.14}$$

$$= \sum_{i=1}^m w_i u(z_j)^{-1}(ue_i)(z_j) \tag{4.15}$$

$$= \sum_{i=1}^m w_i u(z_j)^{-1} \left(\sum_{l=1}^m \overline{\alpha_l}(ue_i)(z_l) \right) \tag{4.16}$$

$$= \sum_{i=1}^m w_i u(z_j)^{-1} \overline{\alpha_i} u(z_i) \tag{4.17}$$

$$= u(z_j)^{-1} \sum_{i=1}^m w_i \overline{\alpha_i} u(z_i) \tag{4.18}$$

$$= w_j u(z_j)^{-1} \sum_{i=1}^m \overline{\alpha_i} u(z_i) \tag{4.19}$$

$$= w_j u(z_j)^{-1} u(z_j) = w_j. \tag{4.20}$$

The matrix case follows easily. $\square$

With this fact in place, we have completed the proof of our interpolation theorem. While Theorem 4.1.1 applies to a fairly broad class of algebras, we do not in general have a way to describe the spaces $\mathcal{M}_u = [\mathcal{A}u]$. At this point we see the value of being able to classify invariant subspaces. After all, the cyclic subspace $[\mathcal{A}u]$ is invariant for the algebra $\mathcal{A}$. In the case of the algebra H_B^∞ we can describe $\mathcal{M}_u$ quite explicitly.

Proposition 4.1.6. *Let B be an inner function and u an outer function. The cyclic subspace $[H_B^\infty u] = W \oplus BH^2$, where W is one-dimensional.*

Proof. If $u = v \oplus Bw$, then $[H_B^\infty u] = [v] \oplus BH^2$. Note that v is non-zero, since the function u is outer and B is inner. In fact, v and B cannot share a common inner factor. This follows from the fact that $\gcd(v, B)$ divides u and u is outer. The inclusion $[H_B^\infty u] \subseteq [v] \oplus BH^2$ is straightforward. Conversely, let

$$f = v \oplus Bh \in ([v] \oplus BH^2) \ominus [Au]. \tag{4.21}$$

Since $f \perp [H_B^\infty u]$ we see that $0 = \langle v + Bh, Bgu \rangle = \langle h, gu \rangle$ for all $g \in H^\infty$. Hence, $h \perp [H^\infty u] = H^2$ which yields $h = 0$. Now,

$$0 = \langle v, u \rangle = \langle v, v + Bw \rangle = \langle v, v \rangle = \|v\|^2. \tag{4.22}$$

which implies $v = 0$. $\square$

Let $v \in H^2 \ominus BH^2$ with $\|v\|_2 = 1$, let $H_v^2 := [v] \oplus BH^2$ and let K^v be the kernel function for the subspace H_v^2. We have

$$K^v(z,w) = v(z)\overline{v(w)} + \frac{B(z)\overline{B(w)}}{1 - z\overline{w}}. \tag{4.23}$$

Hence, for the algebra H_B^∞ we have an interpolation condition where the kernels are indexed by the family of one-dimensional subspaces of $H^2 \ominus BH^2$. If B is finite, say with m zeroes counting multiplicity, then the one-dimensional subspaces of $H^2 \ominus BH^2$ are parameterized by unit vectors in $H^2 \ominus BH^2$, where we identify two vectors that differ only by a unimodular scalar factor. Hence, there is a natural identification of these spaces with the complex projective m-sphere $P^m(\mathbb{C})$. In the case $B = z^2$ we recover [15, Theorem 1.1].

Corollary 4.1.7 ([15, Theorem 1.1])**.** *Let $z_1, \ldots, z_n \in \mathbb{D}$ and $w_1, \ldots, w_n \in \mathbb{C}$. For $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$, let $H^2_{\alpha,\beta} = [\alpha + \beta z] \oplus z^2 H^2$. Let $K^{\alpha,\beta}$ denote the reproducing kernel for the space $H^2_{\alpha,\beta}$. There exists a function $f \in H_1^\infty$ such that $\|f\|_\infty \le 1$ and $f(z_j) = w_j$ if and only if*

$$\left[(1 - w_i\overline{w_j})K^{\alpha,\beta}(z_i, z_j)\right]_{i,j=1}^n \ge 0 \tag{4.24}$$

for all α, β as above and $\alpha \neq 0$.

Proof. This result follows immediately from the fact that the one-dimensional subspaces of $H^2 \ominus z^2H^2$ are of the form $[\alpha + \beta z]$ with $|\alpha|^2 + |\beta|^2 = 1$. The condition that $\alpha \neq 0$ is a consequence of the fact that the cyclic subspace $[H_1^\infty u]$ contains the outer function u and therefore not all functions in $[\alpha + \beta z]$ can vanish at the origin. $\square$

4.2 Interpolation in H^∞_Γ

To begin this section we prove a Nevanlinna-Pick interpolation theorem for the space H^∞_Γ. This result is a generalization of Abrahamse's theorem. Our group Γ is Fuchsian and Abrahamse's theorem is the special case where the group Γ is the group of deck transformations of a multiply connected region R.

The result is a consequence of the distance formula in Theorem 3.2.10. The proof of the interpolation theorem for H^∞_Γ follows from the distance formula just as the Theorem 4.1.1 followed from Theorem 3.2.8.

Theorem 4.2.1 (Nevanlinna-Pick interpolation in H^∞_Γ)**.** *Let Γ be a Fuchsian group such that the defect space N is finite dimensional. Let $z_1, \ldots, z_n \in \mathbb{D}$ and $w_1, \ldots, w_n \in \mathbb{C}$. There exists a function $f \in H^\infty_\Gamma$ with $\|f\|_\infty \leq 1$ such that $f(z_j) = w_j$ if and only if*

$$[(1 - w_i\overline{w_j})K^u(z_i, z_j)]_{i,j=1}^n \geq 0 \tag{4.25}$$

for all $\chi \in \hat{\Gamma}$ and all outer functions $u \in H^2_\chi$, where K^u is the kernel function of $[H^\infty_\Gamma u]$.

When the group Γ is the group of deck transformations of a multiply connected region the result of Abrahamse and Douglas [2], which in turn depends on the projection constructed by Forelli in [20], shows that the subspace $[H^\infty_\Gamma u] = H^2_\chi$, whenever $u \in H^2_\chi$ is outer. In this case, the matrices are really indexed by the elements of $\hat{\Gamma}$. We now know that this result is true for Fuchsian groups with finite-dimensional defect space N. However, the answer to the more general invariant subspace theorem is still open.

Question Is every subspace $\mathcal{M}$ of H^2_χ that is invariant for H^∞_Γ of the form ϕH^2_σ for some $\sigma \in \hat{\Gamma}$ and some character automorphic inner function ϕ?

We will now take a different approach to the study of the interpolation problem for H^∞_Γ. The approach that we describe works only for amenable groups. The original goal of this was to see whether a proof of the interpolation theorem for the annulus $\mathbb{A}$ could be given that did not depend on duality arguments or function theory. Perhaps this would cast more light on the interpolation problem for $H^\infty(\mathbb{A})$. Ultimately, this proved not to work. Nonetheless, a couple of interesting examples have emerged from this effort. It seems worthwhile to look at what happens to the interpolation problem in the spaces H^∞_Γ, when Γ is an amenable group. The term *amenable* means that there exists an invariant state on $\ell^\infty(\Gamma)$.

The focus of this section is examples. So far, we have assumed for the group Γ that H^∞_Γ is non-trivial. We are now more interested in what structure can be deduced about the algebra H^∞_Γ from our knowledge about the group Γ. One of our examples will be the group of deck transformations associated with a covering of an annulus by the disk. Another example will be isomorphic to the free product of the group $\mathbb{Z}_2$ with itself. This latter group has fixed points and cannot be the group of deck transformations for a Riemann surface covering map. A third example is provided by the cyclic subspace generated by a rotation of the disk through an angle $\dfrac{2\pi}{m}$, where $m \in \mathbb{N}$. These three examples are "prototypes" for an amenable Fuchsian group. By this we mean that any amenable Fuchsian group Γ is isomorphic to one of the examples we are about to describe [30].

The simplest example of an amenable Fuchsian group is the one generated by the automorphism $\rho(z) = e^{2\pi i/N} z$. We will denote this group $\Gamma^{(0)}$ and note that $\Gamma^{(0)}$ is isomorphic to $\mathbb{Z}_N$. The space $H^\infty_{\Gamma^{(0)}}$ consists of elements in H^∞ of the form $\sum_{j=0}^\infty a_j z^{jN}$. When viewed as a subalgebra of the multiplication operators on H^2, $H^\infty_{\Gamma^{(0)}}$ is the weak*-closed algebra generated by S^N, where S is the unilateral shift.

To get our second example we consider a covering of the annulus by the disk. Let $\mathbb{A}$ be the annulus with inner radius r and outer radius R. The strip $\mathcal{S} := \{z : 0 < \Re(z) < \pi\}$ is easily seen to be a covering space for the annulus. The covering map is given by $E(z) = \exp\left(\frac{z}{\pi}\log\left(\frac{r}{R}\right)\right)$. To see that the disk and the strip are conformally equivalent we describe a sequence of maps. The function $z \mapsto \dfrac{1+z}{1-z}$ maps the disk onto the right half-plane $\Re(z) > 0$. Multiplication by i maps the right half-plane to the upper half-plane and the principal branch of the logarithm log maps the upper half-plane onto the strip $\{z : 0 < \Im(z) < \pi\}$. Multiplication by $-i$ maps this strip onto $\mathcal{S} := \{z : 0 < \Re(z) < \pi\}$. Let q denote the composition of the maps just described,

$$q(z) = -i \log\left(i\frac{1+z}{1-z}\right). \tag{4.26}$$

The function $E : S \to \mathbb{A}$ is surjective. Let $c = \dfrac{2\pi^2 i}{\log(R) - \log(r)}$ and let $\tau : \mathcal{S} \to \mathcal{S}$ be defined by $\tau(z) = z + c$. It is easy to see that $E \circ \tau = E$ and that no other biholomorphic map of $\mathcal{S}$ onto itself has this property except an integer power of τ. Thus, τ is the generator of the group of deck transformations. The map q is biholomorphic and so we see that $p = E \circ q$ is a covering map of the disk onto $\mathbb{A}$. If γ is the automorphism of the disk that generates the group of deck transformations, then $\gamma^{(n)} = q^{-1} \circ \tau^{(n)} \circ q$ for all $n \in \mathbb{Z}$. It is not hard to check that $\gamma(z) = \dfrac{z-a}{1-az}$

with $a \in (0,1)$. In this case we obtain an infinite cyclic group, which we denote $\Gamma^{(1)}$, generated by $\gamma(z) = \dfrac{z-a}{1-az}$.

The third example is generated by $\beta(z) = -z$ and $\gamma(z) = \dfrac{a-z}{1-az}$ with $a \in (0,1)$. Denote by $\Gamma^{(2)}$ the group generated by β and γ. Every element of $\Gamma^{(2)}$ can be identified with a word in β and γ. Since β and γ have order 2, the elements of $\Gamma^{(2)}$ are words in β and γ with the property that every word is an alternating string of β's and γ's. Hence, there are precisely two words of each length, distinguished by the "letter" they begin with. Set $\alpha = \beta\gamma$ and note $\alpha(z) = \dfrac{z-a}{1-az}$ and so $\Gamma^{(2)}$ contains $\Gamma^{(1)}$. A word in $\Gamma^{(2)}$ is of the form α^m or $\alpha^m\beta$ where $m \in \mathbb{Z}$. Since $\beta(0) = 0$ we see that $\alpha^m(0) = \alpha^m\beta(0)$. If $\alpha^m(0) = 0$, then Lemma 4.2.3 shows $m = 0$ and so α^m is not the identity map for $m \neq 0$. It follows that $\alpha^m\beta$ is also not the identity map for $m \neq 0$. We have shown that $\Gamma^{(2)} \cong \mathbb{Z}_2 * \mathbb{Z}_2$, the free product of $\mathbb{Z}_2$ with itself, which is an amenable group. In fact, this is the only non-abelian free product that is an amenable group. Both these statements are well known. However, we lack a reference and so we provide a short proof of the fact that $\mathbb{Z}_2 * \mathbb{Z}_2$ is amenable.

Proposition 4.2.2. *The free product $G = \mathbb{Z}_2 * \mathbb{Z}_2$ is amenable.*

Proof. Set $\alpha = \beta\gamma$ and note that any word in G can be expressed in the form α^k or $\gamma\alpha^k$, where $k \in \mathbb{Z}$. The cyclic group $H = \langle\gamma\rangle$ is an abelian, hence amenable, subgroup of G. Since γ has order 2, the subgroup H has index 2 in G. Hence, H is normal and the quotient G/H is of order 2. Since the quotient is amenable and H is amenable, it follows from a "2 out of 3" principle [32, Proposition 0.16] that G is amenable. $\square$

One of the ways in which our examples differ from deck transformation groups is the presence of fixed points and torsion.

In the simple cases that we deal with we will be able to reverse our point of view. Instead of assuming that H^∞_Γ is non-trivial, we will explicitly show that the spaces H^∞_Γ are non-trivial. These calculation will involve showing that the Blaschke product for the orbit $\Gamma(0)$ converges and computing the corresponding character. The space $H^\infty_{\Gamma(0)}$ is clearly non-trivial. If we use the fact that $\Gamma^{(1)}$ is a group of deck transformations and identify $H^\infty_{\Gamma(1)}$ with $H^\infty(\mathbb{A})$, then we know that $H^\infty_{\Gamma(1)}$ is non-trivial. However, we will provide a direct proof of the non-triviality of $H^\infty_{\Gamma(1)}$. It is not immediately clear that $H^\infty_{\Gamma(2)}$ is non-trivial. In fact, the elements of $H^\infty_{\Gamma(2)}$ are exactly the functions in $H^\infty_{\Gamma(1)}$ that satisfy $f(z) = f(\beta(z)) = f(-z)$. We will call such a function *even*. The fact that $H^\infty_{\Gamma(2)}$ is non-trivial will also follow from our proof that $H^\infty_{\Gamma(1)}$ is non-trivial.

Lemma 4.2.3. *Let*

$$\gamma(z) = \frac{z-a}{1-az}, \tag{4.27}$$

where $a \in (-1,1)$ and set

$$a_n = \frac{(1+a)^n - (1-a)^n}{(1+a)^n + (1-a)^n}, n \geq 1. \tag{4.28}$$

We have,

$$\gamma^{(n)}(z) = \frac{z-a_n}{1-a_n z}, \tag{4.29}$$

and

$$\gamma^{(-n)}(z) = \frac{z+a_n}{1+a_n z}, \tag{4.30}$$

for $n \geq 1$.

Proof. Note that $\gamma(0) = -a = -a_1$. Now suppose that $\gamma^{(n)}(0) = -a_n$ and consider

$$\gamma^{(n+1)}(0) = \frac{\gamma^{(n)}(0) - a}{1 - a\gamma^{(n)}(0)} \tag{4.31}$$
$$= \frac{\dfrac{(1-a)^n - (1+a)^n}{(1-a)^n + (1+a)^n} - a}{1 - a\dfrac{(1-a)^n - (1+a)^n}{(1-a)^n + (1+a)^n}} \tag{4.32}$$
$$= \frac{(1-a)^n - (1+a)^n - a(1-a)^n - a(1+a)^n}{(1-a)^n + (1+a)^n - a(1-a)^n + a(1+a)^n} \tag{4.33}$$
$$= \frac{(1-a)(1-a)^n - (1+a)(1+a)^n}{(1-a)(1-a)^n + (1+a)(1+a)^n} \tag{4.34}$$
$$= -a_{n+1}. \tag{4.35}$$

Now,

$$\gamma^{(-1)}(z) = \frac{z+a}{1+az} = \frac{z-(-a)}{1-(-a)z} \tag{4.36}$$

and so

$$\gamma^{(-n)}(0) = \frac{(1-(-a))^n - (1-a)^n}{(1-(-a))^n + (1-a)^n} \tag{4.37}$$
$$= \frac{(1+a)^n - (1-a)^n}{(1+a)^n + (1-a)^n} \tag{4.38}$$
$$= a_n. \tag{4.39}$$

For an automorphism $\phi(z) = \lambda\dfrac{a-z}{1-\overline{a}z}$, $a = \phi^{(-1)}(0)$ and $\lambda = \dfrac{\phi(0)}{\phi^{-1}(0)}$. If

$$\gamma^{(n)}(z) = \lambda_n\frac{\gamma^{(-n)}(0) - z}{1 - \overline{\gamma^{(-n)}(0)}z}, \tag{4.40}$$

then $\lambda_n = \dfrac{-a_n}{a_n} = -1$ and $\gamma^{(-n)}(0) = a_n$. So $\gamma^{(n)}(z) = \dfrac{z - a_n}{1 - a_n z}$ for $n \geq 1$ and on computing the inverse we get $\gamma^{(-n)}(z) = \dfrac{z + a_n}{1 + a_n z}$ for $n \geq 1$. □

Lemma 4.2.4. *The Blaschke sum* $\sum_{\gamma \in \Gamma^{(1)}} (1 - |\gamma(0)|)$ *converges.*

Proof. We have that the elements of $\Gamma^{(1)}$ are of the form $\gamma^{(n)}$ where $\gamma(z) = \dfrac{z-a}{1-az}$ with $a \in (0,1)$ and $n \in \mathbb{Z}$. From the previous lemma we have

$$1 - a_n = \frac{2(1-a)^n}{(1+a)^n + (1-a)^n} \leq 2\left(\frac{1-a}{1+a}\right)^n. \tag{4.41}$$

The latter series is geometric and so the sum above converges. □

One consequence of this result is that the Blaschke product for the orbit

$$\Gamma^{(1)}(0) = \{\gamma^{(n)}(0) \,:\, n \in \mathbb{Z}\} = \{0, a_n, -a_n \,:\, n \geq 1\} \tag{4.42}$$

is convergent and we will show that $B \circ \gamma = -B$ and $B(-z) = -B(z)$. For $n \in \mathbb{Z}$, let ϕ_n denote the simple Blaschke factor at $\gamma^{(n)}(0)$. For $n \geq 1$, the simple Blaschke factor at $a_n = \gamma^{(-n)}(0)$ is given by

$$\frac{|a_n|}{a_n} \frac{a_n - z}{1 - a_n z} = -\frac{z - a_n}{1 - a_n z} = -\gamma^{(n)}(z). \tag{4.43}$$

The factor at $-a_n = \gamma^{(n)}(0)$ can be computed similarly and is

$$\frac{|-a_n|}{-a_n} \frac{-a_n - z}{1 + a_n z} = \frac{z + a_n}{1 + a_n z} = \gamma^{(-n)}(z). \tag{4.44}$$

The factor at 0 is z. Our calculation shows that

$$\phi_n = \begin{cases} \gamma^{(-n)} & n \geq 1 \\ z & n = 0 \\ -\gamma^{(-n)} & n \leq -1 \end{cases} . \tag{4.45}$$

Let

$$B_N := \prod_{|j|\leq N} \phi_j = (-1)^N \prod_{|j|\leq N} \gamma^{(j)}. \tag{4.46}$$

Composing with γ and multiplying by $\gamma^{(-N)}$ we get,

$$\gamma^{(-N)}(B_N \circ \gamma) = \gamma^{(-N)}(-1)^N \prod_{|j|\leq N} \gamma^{(j+1)} \tag{4.47}$$

$$= ((-1)^N \prod_{|j|\leq N} \gamma^{(j)})\gamma^{(N+1)} \tag{4.48}$$

$$= B_N \gamma^{(N+1)}. \tag{4.49}$$

Hence, $\gamma^{(-N)}(z)B_N(\gamma(z)) = \gamma^{(n+1)}(z)B_N(z)$. Since $a_n \to 1$ as $n \to \infty$ we get that $\gamma^{(n)}(z) \to -1$ and $\gamma^{(-n)}(z) \to 1$ as $n \to \infty$. Hence, on taking the limit in N we get that $B(\gamma(z)) = -B(z)$.

For $n \in \mathbb{Z}$ we have $\gamma^{(n)}(-z) = -\gamma^{(-n)}(z)$. Hence, $B_N(-z) = (-1)^{2N+1}B_N(z) = -B_N(z)$ and $B(-z) = -B(z)$.

Proposition 4.2.5. *The algebra $H^\infty_{\Gamma^{(2)}}$ is non-trivial.*

Proof. Note that B^2 is even and invariant under γ. $\square$

The stabilizer subgroup $\Gamma^{(2)}_0$ consists of the identity and the automorphism β.

Hence, the Blaschke associated with the group $\Gamma^{(2)}$ is B^2.

It follows from Corollary 2.2.9 that $\{B^{2n} : n \geq 0\}$ is an orthonormal basis for $H^2_{\Gamma^{(2)}}$. Since B^2 is inner we see that $H^\infty_{\Gamma^{(2)}}$ is really the span of the powers of an inner function. The interpolation theory for spaces generated by a single inner function is quite simple as Theorem 4.2.8 shows. To prove Theorem 4.2.8 we require a few preliminary results.

Lemma 4.2.6. *If ϕ is an inner function, then $\overline{\phi(\mathbb{D})} = \overline{\mathbb{D}}$.*

Proof. The operator of multiplication by ϕ on H^2 is isometric but not unitary. By the Wold decomposition the spectrum of M_ϕ is the closed unit disk. Hence, $\overline{\mathbb{D}} = \sigma(M_\phi) = \overline{\phi(\mathbb{D})}$. □

Corollary 4.2.7. *If ϕ is an inner function and $f \in H^\infty$, then $\|f\|_\infty = \|f \circ \phi\|_\infty$.*

Proof. Since $\phi(\mathbb{D}) \subseteq \mathbb{D}$, $\|f \circ \phi\|_\infty \leq \|f\|_\infty$. If $z \in \mathbb{D}$ there exists $\phi(z_n) \in \phi(\mathbb{D})$ such that $\phi(z_n) \to z$ and so $|f(z)| = \lim_{n\to\infty} |f(\phi(z_n))| \leq \|f \circ \phi\|_\infty$. □

Proposition 4.2.8. *Let ϕ be an inner function such that $\phi(0) = 0$. Let $H^2(\phi)$ be the closed span in H^2 of $\{1, \phi, \phi^2, \ldots\}$ and let $H^\infty \circ \phi$ be the weak*-closure in H^∞ of $\{\phi^n : n \geq 0\}$.*

1. *$\{\phi^n : n \geq 0\}$ is an orthonormal basis for $H^2(\phi)$.*
2. *The kernel function for $H^2(\phi)$ is $K^\phi(z, w) = \dfrac{1}{1 - \phi(z)\overline{\phi(w)}}$.*
3. *$f \in H^\infty(\phi)$ if and only if $f \in H^2(\phi) \cap H^\infty$ if and only if $f = g \circ \phi$ for some $g \in H^\infty$ with $\|g\|_\infty = \|f\|_\infty$.*
4. *The multiplier algebra of $H^2(\phi)$ is $H^\infty(\phi)$.*

5. *The space $H^2(\phi)$ is a complete Nevanlinna-Pick space.*

Proof.

1. Since the function ϕ vanishes at the origin we have that ϕ^n is orthonormal and since any element of $H^2(\phi)$ can be approximated by a finite linear combination of the form $a_0 + a_1\phi + \dots a_n\phi^n$ we see that it is a basis.

2. We have
$$K^\phi(z,w) = \sum_{n=0}^{\infty} \phi(z)^n \overline{\phi(w)^n} = \frac{1}{1-\phi(z)\overline{\phi(w)}} \tag{4.50}$$

3. Let us denote by $H^\infty \circ \phi$ the space of functions of the form $\{g \circ \phi \,:\, g \in H^\infty\}$ and by $\mathcal{A}$ the weak*-closed subalgebra of H^∞ spanned by $\{\phi^n \,:\, n \geq 0\}$. The algebra $H^\infty \circ \phi$ is weak* closed and contains $\mathcal{A}$. If $f \in H^2(\phi) \cap H^\infty$, then $f = \sum_{n=0}^\infty a_n \phi^n$ and so $f = (\sum_{n=0}^\infty a_n z^n) \circ \phi$. The sequence a_n is square summable and so $g = \sum_{n=0}^\infty a_n z^n \in H^2$ with $g \circ \phi = f$. However, by Lemma 4.2.7 we get that $\|g\|_\infty = \|f\|_\infty$ and so $g \in H^\infty$. This shows that $H^2(\phi) \cap H^\infty \subseteq H^\infty \circ \phi$. If $f \in H^\infty \circ \phi$, then $f = g \circ \phi$ and we can choose a net of polynomials p_t such that $p_t \to g$ in the weak* topology. Composition by ϕ is weak* continuous and so $p_t \circ \phi \to f$ in the weak* topology and $\mathcal{A} = H^\infty \circ \phi$. Finally note, since $H^2(\phi)$ is closed in H^2, that the space $H^2(\phi) \cap H^\infty$ is weak* closed in H^∞ and contains ϕ^n for all $n \geq 0$. This proves that $\mathcal{A} \subseteq H^2(\phi) \cap H^\infty$.

4. Since $1 \in H^2(\phi)$ and multipliers must be bounded we see that $\mathrm{mult}(H^2(\phi)) \subseteq H^2(\phi) \cap H^\infty = H^\infty(\phi)$. On the other hand, it is clear from $H^\infty(\phi) =$

$H^\infty \circ \phi$ that $H^\infty(\phi) \subseteq \mathrm{mult}(H^2(\phi))$. The equality of norms follows as in Proposition 4.1.2.

5. To see that K^ϕ is a complete Nevanlinna-Pick kernel let us consider n points $z_1, \ldots, z_n$ in the disk and n matrices $W_1, \ldots, W_n$ in M_k. Suppose that the matrix $[(I - W_i W_j^*)K^\phi(z_i, z_j)]_{i,j=1}^n \geq 0$. There exists $F \in M_k(H^\infty)$ such that $\|F\|_\infty \leq 1$ and $F(\phi(z_j)) = W_j$ by the classical Nevanlinna-Pick theorem. The function $\tilde{F} = F \circ \phi$ has the same norm as F and is in $H^\infty(\phi)$. Also $\tilde{F}(z_j) = F(\phi(z_j)) = W_j$.

   On the other hand, suppose that there is a function F in $M_k(H^\infty(\phi))$ such that $F(z_j) = W_j$. We know that $F = G \circ \phi$ for some $G \in M_k(H^\infty)$ with $\|G\|_\infty = \|F\|_\infty$. Hence, $G(\phi(z_j)) = W_j$ and the Pick matrix is positive.

□

Although the proof of Proposition 4.2.8 is elementary we point out that the representation obtained in part (2) implies the result in (5) by the work of Agler-McCarthy, McCullough and Quiggin on complete Nevanlinna-Pick kernels [26, 3, 36]. The inner function ϕ maps the disk $\mathbb{D}$ to the disk. Applying [3, Theorem 3.1] on complete Nevanlinna-Pick kernels we see that the pull-back kernel $K^\phi(z, w) = K(\phi(z), \phi(w))$ is a complete Nevanlinna-Pick kernel for the reproducing kernel Hilbert space associated with K^ϕ. Proposition 4.2.8 shows that the Hilbert space associated with K^ϕ is $H^2(\phi)$.

Applying Theorem 4.2.8 to the space $H^2_{\Gamma(2)}$ we get

Proposition 4.2.9. *Given n points $z_1, \ldots, z_n \in \mathbb{D}$ and n matrices $W_1, \ldots, W_n \in M_k$, there exists a function $F \in M_k(H^\infty_{\Gamma(2)})$ with $\|F\|_\infty \leq 1$ and $F(z_j) = W_j$ if and* *only if*

$$\left[\frac{I - W_i W_j^*}{1 - B(z_i)^2\overline{B(z_j)^2}}\right]_{i,j=1}^n = \left[(I - W_i W_j^*)K^{\Gamma^{(2)}}(z_i, z_j)\right]_{i,j=1}^n \geq 0, \tag{4.51}$$

where B is the Blaschke product for the orbit $\Gamma^{(2)}(0) = \Gamma^{(1)}(0)$ and $K^{\Gamma^{(2)}}$ is the kernel for the reproducing kernel Hilbert space $H^2_{\Gamma^{(2)}}$.

So far we have not discussed the group $\Gamma^{(0)}$. However, note that $H^2_{\Gamma^{(0)}}$ is also of the form $H^2(\phi)$ for the inner function $\phi = z^N$.

We have not as yet used the fact that Γ is amenable. Since the algebra H^∞ is a dual space, we have available to us a weak* averaging argument. This will allow us to build a projection from H^∞ onto H^∞_Γ.

We first make some general comments about group actions on Banach spaces. Let Γ be a group and X be a Banach space. By an action of Γ on X we mean a homomorphism θ from Γ into the set of isometric isomorphisms of X. For $\gamma \in \Gamma$ we denote $\theta(\gamma)$ by γ and the context will make clear whether we are referring to the group element or the linear map. If X is a dual space then we define a *weak*-action* to be an action θ of Γ on X with the additional property that γ is weak* continuous for all $\gamma \in \Gamma$. Given an action θ of Γ we define a dual action on X^* by $\theta^*(\gamma) = (\gamma^*)^{-1}$. Similarly given a weak*-action on X^* we see that it arises from an action on X. If Γ acts on X then we denote by X_Γ the set of all $x \in X$ such that $\gamma(x) = x$ for all $\gamma \in \Gamma$.

When the group Γ is amenable there is another projection from L^∞ onto L^∞_Γ. The following result is well known and we state it for completeness.

Proposition 4.2.10. *If Γ is an amenable group that acts on X, then the map Φ defined by*

$$\Phi(f)(x) = m(f(\gamma(x))) \tag{4.52}$$

is a linear, contractive, idempotent map whose range is $(X^*)_\Gamma$ and $\Phi(\gamma^*(f)) = \Phi(f)$ for all $\gamma \in \Gamma$.

We define a subspace Y of X to be Γ-*invariant* if $\gamma(y) \in Y$ for all $\gamma \in \Gamma$. Note that if Y is Γ-invariant, then the action of Γ on X restricts to an action on Y. To see this we need only prove $\gamma|_Y : Y \to Y$ is surjective and this follows from $Y = \gamma(\gamma^{-1}(Y)) \subseteq \gamma(Y) \subseteq Y$.

Proposition 4.2.11. *If Γ is an amenable group which acts on X and Φ is defined as in Proposition 4.2.10, then we have the following:*

1. *If Y is a Γ-invariant subspace of X, then $Y^\perp \subseteq X^*$ is Γ-invariant and $\Gamma(Y^\perp) = (Y^\perp)_\Gamma$.*
2. *If Y is a Γ-invariant subspace of X^*, then $Y_\perp$ is Γ-invariant.*
3. *If Y is a weak* closed, Γ-invariant subspace of X^*, then $\Phi(Y) = Y_\Gamma$.*

Proof.

1. Let $f \in Y^\perp$ and $y \in Y$. We have, $\gamma^*(f)(y) = f(\gamma(y)) = 0$ and so $\gamma^*(f) \in Y^\perp$. By definition $\Phi(f)(y) = m(f(\gamma(y)) = 0$ and so $\Phi(f) \in Y^\perp$.
2. Let $f \in Y$ and $y \in Y_\perp$. We have $f(\gamma(y)) = \gamma^*(f)(y) = 0$ and so $\gamma(y) \in Y_\perp$.
3. Since Y is weak* closed we have that $Y = (Y_\perp)^\perp$. As Y is Γ-invariant so is $Y_\perp$ and we get $\Phi(Y) = \Phi((Y_\perp)^\perp) = ((Y_\perp)^\perp)_\Gamma = Y_\Gamma$.

$\square$

The case we are interested in is $X = L^\infty$. In this case, Φ is a unital positive map and is therefore completely contractive.

If $Y \subseteq X^*$ is Γ-invariant, then so is its weak*-closure. To see this note that $Y_\perp$ is Γ-invariant and so is $\overline{Y}^{\text{weak}^*} = (Y_\perp)^\perp$. If we apply this last proposition to the weak*-closed, Γ-invariant subalgebra H^∞ of L^∞ then we see that the projection above maps H^∞ to H^∞_Γ.

In some ways this new projection is an improvement on the one in [20], since it avoids the defect space N, preserves analytic structure and is completely contractive. The following result shows, in the case of H^∞, that Φ is in some sense the natural projection to consider.

Proposition 4.2.12. *Let $f \in H^\infty$ and let the power series expansion of $f \circ \gamma$ be given by $\sum_{n=0}^\infty a_n(\gamma) z^n$. For each n, $(a_n(\gamma))_{\gamma\in\Gamma} \in \ell^\infty(\Gamma)$. If $b_n = m(a_n(\gamma))$, then*

$$\Phi(f)(z) = m(f(\gamma(z))) = \sum_{n=0}^\infty b_n z^n. \tag{4.53}$$

Proof. If k_z is the Szegö kernel at z, then

$$\Gamma(f)(z) = \langle \Phi(f), k_z \rangle = m(\langle f \circ \gamma, k_z \rangle) = m(f(\gamma(z))). \tag{4.54}$$

We have, $|a_n(\gamma)| = |\langle f \circ \gamma, \chi_n \rangle| \leq \|f\|_\infty$ and so for fixed n, $(a_n(\gamma))_{\gamma\in\Gamma} \in \ell^\infty(\Gamma)$. Now,

$$\langle \Phi(f), z^n \rangle = m(\langle f \circ \gamma, z^n \rangle) = m(a_n(\gamma)) = b_n. \tag{4.55}$$

Hence, $\Phi(f)(z) = \sum_{n=0}^\infty b_n z^n$. $\square$

For the special of H^∞, there is a second averaging result that yields the same projection.

Theorem 4.2.13. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ and suppose that $\mathcal{A} = \mathcal{A}'$. If Γ is an amenable group that acts on $\mathcal{A}$, then there exists a linear, contractive, idempotent map $\Phi : \mathcal{A} \to \mathcal{A}_\Gamma$. If $\mathcal{A}$ is selfadjoint, then so is Φ.*

Proof. Let $A \in \mathcal{A}$ and $f_1, f_2 \in \mathcal{H}$. Define a map $s_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by

$$s_A(f_1, f_2) := m(\langle \gamma(A) f_1, f_2 \rangle) \tag{4.56}$$

The map s_A is a bounded sesquilinear form on $\mathcal{H}$ with $\|s_A\| \leq \|A\|$. Therefore, there exists a unique operator $\Phi(A) \in B(\mathcal{H})$ such that

$$\langle \Phi(A) f_1, f_2 \rangle = m(\langle \gamma(A) f_1, f_2 \rangle). \tag{4.57}$$

Let $A \in \mathcal{A}$ and note that

$$\begin{aligned}
\langle \Phi(A) T f_1, f_2 \rangle &= m(\langle \gamma(A) T f_1, f_2 \rangle) && (4.58)\\
&= m(\langle T \gamma(A) f_1, f_2 \rangle) && (4.59)\\
&= m(\langle \gamma(A) f_1, T^* f_2 \rangle) && (4.60)\\
&= \langle \Phi(A) f_1, T^* f_2 \rangle && (4.61)\\
&= \langle T \Phi(A) f_1, f_2 \rangle . && (4.62)
\end{aligned}$$

Thus, $\Phi(A) \in \mathcal{A}' = \mathcal{A}$. The remaining properties are easy to verify. □

It is easy to check, in the case $\mathcal{H} = H^2$ and $\mathcal{A} = H^\infty$, that this agrees with the projection in Proposition 4.2.10.

Proposition 4.2.14. *If Φ_j is the projection from H^∞ onto $H^\infty_{\Gamma^{(j)}}$ for $j = 1, 2$, then $\Phi_j(A(\mathbb{D}))$ is the set of constant functions.*

Proof. Since $\gamma^{(k)}(z) \to -1$ as $k \to \infty$ and 1 as $k \to -\infty$ we have that,

$$m(((\gamma^{(k)}(z))^n)_{k\in\mathbb{Z}}) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \\ 1 & \text{if } k \text{ is even} \end{cases} . \tag{4.63}$$

Therefore, Φ_j maps z^n to a constant and so $A(\mathbb{D})$ is also mapped to the constant functions. □

Corollary 4.2.15. *There are no non-constant $\Gamma^{(j)}$-invariant functions in $A(\mathbb{D})$ for $j = 1, 2$.*

Proof. If $f \in A(\mathbb{D})$ and is fixed by Φ_j, then $f = \Phi_j(f)$ is constant. □

Corollary 4.2.16. *Φ_j is not weak* continuous.*

Proof. This is a consequence of the fact that the trigonometric polynomials are weak* dense in H^∞ and that $H^\infty_{\Gamma^{(j)}}$ contains non-constant functions. □

The averaging argument is known to fail for regions of connectivity greater than 2, see Barrett [12]. Since interpolation is an isometric theory the existence of a contractive projection suggests, at least in the case of amenable groups, that we could approach the problem of interpolation in H^∞_Γ through a related problem on the disk. We will show that while this approach has the positive aspect of providing a fairly simple matrix positivity condition, the condition does not seem refined enough to distinguish the interpolation theory of $H^\infty_{\Gamma(1)}$ and $H^\infty_{\Gamma(2)}$.

Let K denote the Szegö kernel on the disk, i.e, the function $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ given by $K(z, w) = \dfrac{1}{1 - z\overline{w}}$. By a *weak solution* to the interpolation problem we mean a matrix valued function $F = [f_{i,j}] \in M_m(H^\infty)$ such that $\|F\|_\infty \leq 1$, $F(\gamma(z_l)) = W_l$ for all $l = 1, \dots, n$ and $\gamma \in \Gamma$. By a *strong solution* we mean an $F = [f_{i,j}] \in M_m(H^\infty_\Gamma)$ such that $\|F\|_\infty \leq 1$ and $f(z_l) = W_l$ for $l = 1, \dots, n$. Clearly every strong solution is a weak solution. If we denote by $\mathcal{W}$ the set of weak solutions, then it is clear, provided that $\mathcal{W} \neq \emptyset$, that $\mathcal{W}$ is a convex, weak*-compact subset of the unit ball of $M_k(H^\infty)$.

Theorem 4.2.17. *Let Γ be an amenable group. If $F \in M_m(H^\infty)$ is weak solution, then $\Phi_m(F)$ is a strong solution.*

Proof. We have $\|\Phi_m(F)\|_\infty \leq \|F\|_\infty \leq 1$, $\Phi_m(F) \in M_m(H^\infty_\Gamma)$ and

$$\Phi_m(F)(z_l) = (m(f_{i,j}(\alpha(z_l)))) = W_l. \tag{4.64}$$

□

If $Q = [q_{i,j}]_{i,j \in J}$ is an infinite matrix, then we write $Q \geq 0$ to mean that every finite square submatrix is positive. This is also known as *formal positivity* but we will suppress the word formal. If Q is already finite, or is the matrix of positive operator on a Hilbert space, then the two notions of positivity coincide. The content of the next result is well known and a proof is provided for completeness.

Proposition 4.2.18. *Let $\{z_j\}_{j\in J}$ be a set of points in $\mathbb{D}$ and $\{W_j\}_{j\in J} \subseteq M_m$. There exists $f \in M_m(H^\infty)$ such that $\|f\|_\infty \leq 1$ and $f(z_j) = W_j$ if and only if the matrix $\left[(I_m - W_i W_j^*)K(z_i, z_j)\right]_{i,j\in J} \geq 0$.*

Proof. Let $\mathcal{F}$ denote the collection of finite subsets of J ordered by inclusion. By Nevanlinna-Pick the positivity of the matrix is equivalent to the existence, for each $F \in \mathcal{F}$, of a function $f_F \in M_m(H^\infty)$ such that $f_F(z_j) = W_j$ for all $j \in F$. Since $\mathcal{F}$ is a directed set and the net $\{f_F\}_{F\in\mathcal{F}}$ is contained in the weak*-compact set $M_m(H^\infty)$ we know that is has a convergent subnet. Let the limit of this subnet be $f \in M_m(H^\infty)$. Since weak* limits preserve point values we get that $f(z_j) = W_j$. □

If we take the set of points in Proposition 4.2.18 to be the orbits of n points $z_1, \dots, z_n$ under the group Γ, then we obtain the following result.

Corollary 4.2.19. *Let $P(z,w)$ denote the infinite kernel matrix*

$$P(z,w) := \left[K(\alpha(z), \beta(w))\right]_{\alpha,\beta\in\Gamma}. \tag{4.65}$$

A weak solution exists if and only if $\left[(I_m - W_i W_j^*)P(z_i, z_j)\right]_{i,j=1}^n \geq 0$

We know that this condition in Proposition 4.2.19 is equivalent to the condition set forth by Abrahamse for the annulus $\mathbb{A}$. It would be interesting to have a direct proof that the matrix condition in Proposition 4.2.19 is equivalent to Abrahamse's condition that the matrices A_λ, $\lambda \in \mathbb{T}$, are all positive semi-definite.

Chapter 5

The C^*-envelope of $H_B^\infty/\mathcal{I}$.

5.1 Matrix-valued interpolation

There is a matrix-valued generalization of the Nevanlinna-Pick problem where the scalars $w_1, \ldots, w_n \in \mathbb{D}$ are replaced by $k \times k$ matrices $W_1, \ldots, W_n$ in the unit ball of M_k. The interpolating function F is now a bounded, matrix-valued, analytic map of $\mathbb{D}$ into the unit ball of M_k. This is equivalent to the existence of F in the unit ball of the Banach algebra $M_k(H^\infty)$ where the norm is given by

$$\|F\| := \sup\{\|F(z)\|_{M_k} \, : \, z \in \mathbb{D}\}. \tag{5.1}$$

The matrix-valued theorem for the disk is the same, aside from the obvious changes.

Theorem 5.1.1 (matrix-valued Nevanlinna-Pick theorem). *Let $z_1, \ldots, z_n \in \mathbb{D}$ and let $W_1, \ldots, W_n \in M_k$. There exists a matrix-valued analytic function $F : \mathbb{D} \to M_k$*

with $\|F\|_\infty \leq 1$ *and* $F(z_j) = W_j$ *for* $j = 1, \ldots, n$ *if and only if*

$$\left[\frac{1 - W_i W_j^*}{1 - z_i \overline{z_j}}\right]_{i,j=1}^{n} \geq 0 \tag{5.2}$$

Let ϕ be an inner function and let $\pi : H^\infty/\phi H^\infty \to B(H^2 \ominus \phi H^2)$ be given by $\pi(f + \phi H^\infty) = PM_f P$, where P is the orthogonal projection of H^2 onto $H^2 \ominus \phi H^2$. Sarason's generalized interpolation result shows that the representation π is an isometry and the special case where ϕ is the Blaschke product for the n points $z_1, \ldots, z_n$ yields the Nevanlinna-Pick theorem. The matrix-valued Nevanlinna-Pick theorem is equivalent to the fact that π is completely isometric.

Shortly after Abrahamse's original proof of the Nevanlinna-Pick theorem for multiply connected domains, Ball [10] proved the matrix-valued generalization. To deal with matrix-valued interpolation one needs to consider the appropriate analogue of the character automorphic spaces. To fix ideas consider an m-holed domain R. The fundamental group of this region is $\mathbb{F}_m$ and the characters of $\mathbb{F}_m$ are just the one-dimensional unitary representations of $\mathbb{F}_m$. For the M_p-valued problem we need to consider unitary representations of the free group $\mathbb{F}_m$ on $\mathbb{C}^p$. Note that there is a natural action of the p-dimensional unitary group $\mathcal{U}_p := \mathcal{U}(\mathbb{C}^p)$ on the space of all p-dimensional unitary representations $\mathrm{Hom}(\mathbb{F}_m, \mathbb{C}^p)$. Given a representation $\pi \in \mathrm{Hom}(\mathbb{F}_m, \mathbb{C}^p)$ and a unitary $U \in \mathcal{U}_p$, the action is given by $\pi(\cdot) \mapsto U^*\pi(\cdot)U$. Two unitary representations are identified if and only if they are unitarily equivalent. Hence, the parameter space is the orbit space of all representations with respect to the natural action of the group $\mathcal{U}_p$. In the scalar-valued case the unitary representations are one-dimensional and the unitary equivalence of representations is the same as equality of representations. Hence, in the scalar case the parameter space is the dual of $\mathbb{F}^m$, namely $\mathbb{T}^m$. In the case of the annulus, the fundamental group is singly generated and any unitary representation is determined by a single $p \times p$ unitary matrix. Any such representation is unitarily equivalent to a representation on the $p \times p$ unitary, diagonal matrices. These representations decompose as a direct sum of one-dimensional representations. Therefore, in the case of the annulus it is again enough to consider just the character space $\widehat{\mathbb{F}_1} = \widehat{\mathbb{Z}} = \mathbb{T}$. The Abrahamse-Ball theorem for the annulus reads as follows:

Theorem 5.1.2 (Abrahamse-Ball)**.** *Let $z_1, \dots, z_n \in \mathbb{A}$ and $W_1, \dots, W_n \in M_p$. There exists a function $F \in M_p(H^\infty(\mathbb{A}))$ such that $\|F\|_\infty \leq 1$ and $F(z_j) = W_j$ if and only if the $pn \times pn$ matrices*

$$A_\lambda := \left[(1 - W_i W_j^*) K^\lambda(z_i, z_j)\right]_{i,j=1}^n \geq 0 \tag{5.3}$$

for all $\lambda \in \mathbb{T}$.

Here too we can interpret the matrix-valued theorem as a statement about a completely isometric representation of $H^\infty(\mathbb{A})/\mathcal{I}$.

While the test condition in Pick's theorem is a single $np \times np$ matrix, in the Abrahamse-Ball theorem we need to check the positivity of an infinite family of matrices indexed by the torus. Abrahamse conjectured that a dense subset of the torus was required for the existence of a solution to the Pick problem. When z, w are fixed, a theorem of Widom [40] shows that the kernel $K^\lambda(z, w)$ varies continuously with λ and this latter density condition is equivalent to requiring that all the matrices A_λ be positive. The work of Ball and Clancey [11] provided the first partial answer to this question. Their work showed, for the case of the annulus $\mathbb{A}$, that if U is an open subset of $\mathbb{T}$, then there exists $z_1, z_2 \in \mathbb{A}$ and $w_1, w_2 \in \mathbb{D}$ such that matrices $A_\lambda \geq 0$ for all $\lambda \in \mathbb{T} \setminus U$, but there exists $\mu \in U$ such that A_μ is not positive.

Fedorov and Vinnikov [18] provided an understanding of how many parameters, or conditions, are required to guarantee the existence of a solution. The result is most easily understood for the case of an annulus $\mathbb{A}$. Fedorov and Vinnikov showed that once the points $z_1, \ldots, z_n \in \mathbb{A}$ are fixed there exist two points $\lambda, \mu \in \mathbb{T}$ such that the positivity of the matrices A_λ and A_μ guarantees the existence of a scalar-valued solution. The parameters λ and μ depend on the points $z_1, \ldots, z_n \in \mathbb{D}$ and as the points $z_1, \ldots, z_n$ vary, the parameters vary with them, eventually exhausting all points on the circle.

It would seem, from a first glance at Theorem 5.1.2, that the scalar and matrix-valued theories for the annulus is the same, i.e., the points in the torus parametrize the test conditions. The matrix-valued case is quite different. In the matrix-valued case, even when the points $z_1, \ldots, z_n$ are fixed, all the matrix positivity (or at least a dense subset) conditions are required to guarantee a solution to the interpolation problem. This fact is proven in [27].

All the interpolation theorems that we have looked at have shown that a scalar-valued solution exists if and only if every member of certain family of $n \times n$ matrices is positive. In the case of the disk, there is one such matrix. In Abrahamse's theorem for $\mathbb{A}$ there is a family of them indexed by $\mathbb{T}$. For H^∞_B, the indexing set is the set of one-dimensional subspaces of $H^2 \ominus BH^2$. Every time we have such a result, we have, in fact, produced an isometric representation of $\mathcal{A}/\mathcal{I}$.

This representation may or may not be completely isometric, but when it is a complete isometry we obtain the analogous matrix-valued interpolation theorem. There could potentially be many Hilbert spaces on which $\mathcal{A}/\mathcal{I}$ has an isometric, or completely isometric, representation. Every isometric representation can be viewed as a condition, albeit a very abstract condition, for the solution to the scalar Nevanlinna-Pick problem.

For example, Abrahamse's theorem tells us that there is an isometric representation of $H^\infty(\mathbb{A})/\mathcal{I}$ on the direct sum of $\mathcal{K}^\lambda$, where $\mathcal{K}^\lambda$ is the span of the kernel functions for $H^2_\lambda(\mathbb{A})$ at the points $z_1, \ldots, z_n \in \mathbb{A}$. However, the result of Fedorov and Vinnikov shows that we can do better since there is an isometric representation on $\mathcal{K}^\lambda \oplus \mathcal{K}^\mu$ for some $\lambda, \mu \in \mathbb{T}$. In some sense Abrahamse's theorem has an infinite number of test conditions, where in fact just two would have sufficed.

In dealing with the matrix-valued problem McCullough and Paulsen [28] developed a way to decide if one has the best possible interpolation theorem. Loosely speaking when we have the fewest test conditions, we have also found the minimal completely isometric representation of the quotient $\mathcal{A}/\mathcal{I}$. Their approach involves computing the C^*-envelope of the quotient $\mathcal{A}/\mathcal{I}$. For the algebra $H^\infty(\mathbb{A})/\mathcal{I}$, the C^*-envelope is $M_n(C(\mathbb{T}))$ which verifies that the Abrahamse-Ball theorem is really the best one could hope for. This approach to the interpolation problem was used in [15] to show that the scalar interpolation theorem for $\mathbb{C} + z^2 H^\infty$ is no longer true in the matrix-valued setting.

In light of these results we would like to look at the the matrix-valued interpolation problem for the algebra H^∞_B. In the next section we will begin by describing the C^*-envelope approach to the interpolation problem from [28]. After this we will outline the dimension jump phenomenon discovered by Solazzo [38]. Finally, we will compute the C^*-envelope of $H_B^\infty/\mathcal{I}$ for some specific cases.

5.2 The C^*-envelope of $H_B^\infty/\mathcal{I}$.

The C^*-envelope of an operator algebra was defined by Arveson [8]. Loosely speaking the C^*-envelope of an operator algebra $\mathcal{A}$, which is denoted $C_e^*(\mathcal{A})$, is the smallest C^*-algebra on which $\mathcal{A}$ has a completely isometric representation. Arveson's work established the existence of the C^*-envelope in the presence of what are called boundary representations. The existence of the C^*-envelope of an operator algebra was established in full generality by Hamana, whose approach did not have any relation to boundary representations.

Theorem 5.2.1 (Arveson-Hamana)**.** *Let $\mathcal{A}$ be an operator algebra. There exists a C^*-algebra, which is denoted $C_e^*(\mathcal{A})$, such that*

1. *There is a completely isometric representation $\gamma : \mathcal{A} \to C_e^*(\mathcal{A})$.*
2. *Given a completely isometric representation $\sigma : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is a C^*-algebra and $C^*(\sigma(\mathcal{A})) = \mathcal{B}$, there exists an onto $*$-homomorphism $\pi : \mathcal{B} \to C_e^*(\mathcal{A})$ such that $\pi \circ \sigma = \gamma$.*

It is easy to see that the C^*-envelope is essentially unique up to $*$-isomorphism. For a detailed description of the C^*-envelope we refer the reader to [34].

For the algebra H^∞ the C^*-envelope of $H^\infty/\mathcal{I}$ is M_n. In [38] the algebras

$$H^\infty_{a_1,\ldots,a_m} := \{f \in H^\infty \,:\, f(a_1) = \ldots = f(a_m)\}, \tag{5.4}$$

were examined and the following result was obtained.

Theorem 5.2.2 (Sollazo [38])**.** *Let $a_1 = 0$, $a_2 = \frac{1}{2}$ and let $z_1 = 0$, $z_2, z_3 \in \mathbb{D}$ with z_1, z_2, z_3 distinct. The C^*-envelope of the algebra $H_B^\infty/\mathcal{I}$ is M_4.*

Note that the quotient $H_B^\infty/\mathcal{I}$ is a 3-idempotent algebra [33]. When we compare this to the classical case we see there has been a jump in the dimension of the C^*-envelope from 3 to 4.

This *dimension jump* phenomenon has also been observed in [15, Theorem 5.3] for the algebra $\mathbb{C} + z^2H^\infty$. In this section we will show, given certain constraints on the number of zeroes in the Blaschke product B, that a similar result is true for the algebra $H_B^\infty/\mathcal{I}$. The first step in understanding the quotient $H_B^\infty/\mathcal{I}$ is to gain some knowledge about the structure of the ideal $\mathcal{I}$.

We will consider only the case where B is a finite Blaschke product. To fix notation we let $\alpha_1, \ldots, \alpha_p$ be the zeroes of B, we assume that these are distinct and have multiplicity $m_j \geq 1$ and we set $m = m_1 + \ldots + m_p$. We arrange the points $z_1, \ldots, z_n$ so that $B(z_j) = 0$ for $j = 1, \ldots, r$ and $B(z_j) \neq 0$ for $j = r+1, \ldots, n$. Denote by E the Blaschke product for the points $z_1, \ldots, z_n$. It is clear that $\mathcal{I} = H_B^\infty \cap EH^\infty$. Since B is a finite Blaschke product we see that $W \subseteq (H^2 \ominus BH^2) \cap H^\infty$ and $\mathcal{I} = E(W + BH^\infty)$. This can also be seen directly from the fact that $\mathcal{I}$ is invariant under H_B^∞.

Theorem 5.2.3. *Let B be a finite Blaschke product and let $\mathcal{I}$ be the ideal of functions in H_B^∞ that vanish at the n points $z_1, \ldots, z_n$. If $r = 0$, then*

$$\mathcal{I} = E([w] + BH^\infty) \tag{5.5}$$

for some $w \in H^\infty \cap (H^2 \ominus BH^2)$*. If* $r \geq 1$*, then*

$$\mathcal{I} = \mathrm{lcm}(B,E)H^\infty = E(W + BH^\infty), \tag{5.6}$$

where W *is* r*-dimensional.*

Proof. Let $f \in \mathcal{I}$ and write $f = \lambda + Bg \in EH^\infty$, where $g \in H^\infty$. By evaluating at $z_1, \ldots, z_n$ we obtain $\lambda + B(z_j)g(z_j) = 0$.

First, consider the case where $r = 0$. We can write $f = \lambda + B(\sum_{j=1}^n c_j k_{z_j}) + BEh$ for some choice of $c_1, \ldots, c_n \in \mathbb{C}$ and $h \in H^\infty$. Hence, $\lambda + B(\sum_{j=1}^n c_j k_{z_j})$ is 0 at the points $z_1, \ldots, z_n$ and so $\lambda + B(z_i)\sum_{j=1}^n c_j K(z_i, z_j) = 0$ for $i = 1, \ldots, n$. Rewriting this as a linear system we get

$$\begin{bmatrix} B(z_1) & & \\ & \ddots & \\ & & B(z_n) \end{bmatrix} \begin{bmatrix} K(z_1,z_1) & \cdots & K(z_1,z_n) \\ \vdots & & \vdots \\ K(z_n,z_1) & \cdots & K(z_n,z_n) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = -\lambda \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{5.7}$$

Since $r = 0$, this system has a unique solution and the constants $c_1, \ldots, c_n$ can be taken to depend linearly on λ. In this case W is one-dimensional.

If $r \geq 1$, then $\lambda = 0$ and $g(z_j) = 0$ for $j = r+1, \ldots, n$. Hence, $f = B\phi_{z_{r+1}} \cdots \phi_{z_n} h$, $f \in \mathrm{lcm}(B,E)H^\infty$ and $\mathcal{I} \subseteq \mathrm{lcm}(B,E)H^\infty$. The reverse inclusion is straightforward. Let $C = B\overline{\mathrm{gcd}(B,E)} \in H^2$. From

$$\begin{aligned} \mathrm{lcm}(B,E)H^2 &= \mathrm{lcm}(B,E)\left([k_{z_1}, \ldots, k_{z_r}] \oplus \mathrm{gcd}(B,E)H^2\right) && (5.8)\\ &= EB\overline{\mathrm{gcd}(B,E)}\left([k_{z_1}, \ldots, k_{z_r}] \oplus \mathrm{gcd}(B,E)H^2\right) && (5.9)\\ &= E(C[k_{z_1}, \ldots, k_{z_r}] \oplus BH^2), && (5.10) \end{aligned}$$

we see that W is r-dimensional. $\square$

For an outer function u,

$$[\mathcal{I}u] = [\mathrm{lcm}(B,E)H^\infty u] = \mathrm{lcm}(B,E)[H^\infty u] = \mathrm{lcm}(B,E)H^2. \tag{5.11}$$

We have seen that $[H_B^\infty u] = [v] \oplus BH^2$ for some vector v and so

$$\mathcal{K}_u = ([v] \oplus BH^2) \ominus \mathrm{lcm}(B,E)H^2 \tag{5.12}$$

$$= [v] \oplus B(H^2 \ominus \phi_{z_{r+1}} \cdots \phi_{z_n} H^2) \tag{5.13}$$

$$= [v] \oplus B[k_{z_{r+1}}, \ldots, k_{z_n}]. \tag{5.14}$$

The space $\mathcal{K}_u$ has dimension $(n-r)+1$. Note that this is also the dimension of the quotient algebra $H_B^\infty/\mathcal{I}$. Our distance formula says that interpolation is possible if and only if the compression of M_f^* to $[v] \oplus B[k_{z_{r+1}}, \ldots, k_{z_n}]$ is a contraction for all $v \in H^2 \ominus BH^2$.

In the case where one or more of the points $z_1, \ldots, z_n$ is a zero of B, i.e., when $r \geq 1$, the distance of $f \in L^\infty$ from $\mathcal{I}$ is the distance of f from $\mathrm{lcm}(B,E)H^\infty$. This is the case we will examine more closely. The objective will be to show that the scalar-valued result in Theorem 4.1 is not the correct matrix-valued interpolation result.

A basis for $\mathcal{K} = H^2 \ominus \mathrm{lcm}(B,E)H^2$ is given by the vectors

$$\mathcal{E} := \{z^i k_{\alpha_j}^{i+1} \,:\, 1 \leq j \leq p, 0 \leq i \leq m_j - 1\} \cup \{k_{z_{r+1}}, \ldots, k_{z_n}\}. \tag{5.15}$$

We begin by computing the matrix of $M_f^*|_{\mathcal{K}}$ with respect to the basis $\mathcal{E}$. It is an elementary calculation to show, for $f \in H^2$ and $m \geq 0$, that

$$\frac{f^{(m)}(w)}{m!} = \left\langle f, z^m k_w^{m+1} \right\rangle. \tag{5.16}$$

Lemma 5.2.4. *If $f \in H^\infty$, then*

$$M_f^*(z^m k_w^{m+1}) = \sum_{j=0}^{m} \frac{1}{j!} \overline{f^{(j)}(w)} z^{m-j} k_w^{m-j+1}. \tag{5.17}$$

Proof. Let $g \in H^2$ and consider

$$\begin{aligned}
\left\langle g, M_f^* z^m k_w^{m+1} \right\rangle = \left\langle fg, z^m k_w^{m+1} \right\rangle &= \frac{(fg)^{(m)}(w)}{m!} && (5.18)\\
&= \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} f^{(j)}(w) g^{(m-j)}(w) && (5.19)\\
&= \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} f^{(j)}(w)(m-j)! \left\langle g, z^{m-j} k_w^{m-j+1} \right\rangle && (5.20)\\
&= \left\langle g, \sum_{j=0}^{m} \frac{1}{j!} \overline{f^{(j)}(w)} z^{m-j} k_w^{m-j+1} \right\rangle. && (5.21)
\end{aligned}$$

From this, we see that

$$M_f^*(z^m k_w^{m+1}) = \sum_{j=0}^{m} \frac{1}{j!} \overline{f^{(j)}(w)} z^{m-j} k_w^{m-j+1}. \tag{5.22}$$

□

When $f \in H^\infty_B$, Lemma 5.2.4 shows us that the matrix of M_f^* is diagonal with respect to the basis $\mathcal{E}$. The matrix of $M_f^*|_{\mathcal{K}}$ is given by

$$D_f^* = \begin{bmatrix} \overline{f(\alpha_1)}I_{m_1} & & & & & \\ & \ddots & & & & \\ & & \overline{f(\alpha_p)}I_{m_p} & & & \\ & & & \overline{f(z_{r+1})} & & \\ & & & & \ddots & \\ & & & & & \overline{f(z_n)} \end{bmatrix}, \tag{5.23}$$

If we partition the basis $\mathcal{E}$ as in (5.15), then the grammian matrix with respect to this basis has the form

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & P \end{bmatrix}, \tag{5.24}$$

where P is the Pick matrix for the points $z_{r+1}, \ldots, z_n$. Since Q is the grammian matrix of a linearly independent set it is invertible and positive. The matrix Q_1 is $m \times m$, positive and invertible, and the matrix Q_2 is an $m \times (n-r)$ matrix of rank $\min\{m, n-r\}$.

For a function $f \in H^\infty$, Sarason's generalized interpolation shows that the distance of f from the ideal ϕH^∞, i.e., $\|f + \phi H^\infty\|$, is given by the norm of the compression of M_f to $H^2 \ominus \phi H^2$. This distance formula is also valid in the matrix-valued case. Let T be an operator on a finite-dimensional Hilbert space $\mathcal{H}$, of dimension n say, let $\mathcal{E}$ be a Hamel basis for $\mathcal{H}$ and let A be the matrix of T with respect to $\mathcal{E}$. The operator T is a contraction if and only if $I_n - A^*QA \geq 0$, where Q is the grammian for $\mathcal{E}$. Using this last fact, if $f \in H_B^\infty$, then

$$\left\|M_f^*|_{\mathcal{K}}\right\| \leq 1 \iff Q - D_f Q D_f^* \geq 0 \tag{5.25}$$

$$\iff Q^{1/2}(I - Q^{-1/2}D_f Q D_f^* Q^{-1/2})Q^{1/2} \geq 0 \tag{5.26}$$

$$\iff I - Q^{-1/2}D_f Q D_f^* Q^{-1/2} \geq 0 \tag{5.27}$$

$$\iff I - (Q^{-1/2}D_f Q^{1/2})(Q^{-1/2}D_f Q^{1/2})^* \geq 0. \tag{5.28}$$

This induces a completely isometric embedding ρ of $H_B^\infty/\mathcal{I}$ in M_{m+n-r} given by

$$\rho(f) = Q^{-1/2}D_f Q^{1/2}. \tag{5.29}$$

The universal property of the C^*-envelope tells us that $C_e^*(H_B^\infty/\mathcal{I})$ is a quotient of $\mathcal{B} := C^*(\rho(H_B^\infty/\mathcal{I}))$. Since we are dealing with a representation on a finite-dimensional space we know that $\mathcal{B}$ is a direct sum of matrix algebras. In the event that $\mathcal{B} = M_{m+n-r}$ we see that $\mathcal{B} = C_e^*(H_B^\infty/\mathcal{I})$. This follows from the fact that M_{m+n-r} is simple.

Theorem 5.2.5. *Let $r \geq 1$ and let $\mathcal{B}$ be the C^*-subalgebra of M_{m+n-r} generated by the image of ρ. The algebra $\mathcal{B} = M_{m+n-r}$ if and only if $m \leq n - r$.*

Proof. We examine the commutant of $\mathcal{B}$ and show that $\mathcal{B}'$ contains only scalar multiples of the identity. Let $R = Q^{1/2}$ and let $RXR^{-1} \in \mathcal{B}'$.

It is possible to choose $f \in H_B^\infty$ such that $f(\alpha_i) = 1$ for all $1 \leq i \leq p$ and $f(z_j) = 0$ for $r + 1 \leq j \leq n$. Given j, with $r + 1 \leq j \leq n$, it is possible to choose f such that $f(z_j) = 1$, $f(\alpha_i) = f(z_l) = 0$ for $1 \leq i \leq p$ and $l \neq j$. Therefore $\mathcal{B}$ is generated by $R^{-1}E_jR$ where $E_0 := E_{1,1} + \ldots + E_{m,m}$ and $E_j := E_{m+j,m+j}$ for $1 \leq j \leq n - r$.

The matrix $RXR^{-1} \in \mathcal{B}'$ if and only if

$$RXR^{-1}R^{-1}E_jR = R^{-1}E_jRRXR^{-1} \tag{5.30}$$

and

$$RXR^{-1}(R^{-1}E_jR)^* = (R^{-1}E_jR)^*RXR^{-1}. \tag{5.31}$$

This happens if and only if $QXQ^{-1}E_j = E_jQXQ^{-1}$ and $XE_j = E_jX$. These conditions tell us that X and QXQ^{-1} are both block diagonal with 1 block of size $m \times m$ followed by $n-r$ blocks of size 1. Let us write

$$X = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, QXQ^{-1} = \begin{bmatrix} B & 0 \\ 0 & E \end{bmatrix}, \tag{5.32}$$

where D and E are scalar diagonal of size $(n-r)$. We have,

$$\begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & P \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & P \end{bmatrix}. \tag{5.33}$$

This tells us that $PD = EP$, where $P = [p_{i,j}]_{i,j=1}^{(n-r)}$ is the Pick matrix. Since $p_{i,j}d_i = p_{i,j}e_j$ and $p_{i,j}$ are non-zero for $1 \leq i,j \leq (n-r)$, we get $d_i = e_j$ for $1 \leq i,j \leq (n-r)$. Hence, we may assume that $D = E = I_{n-r}$. Now comparing the off-diagonal entry we see that $Q_2^*A = Q_2^*$, $BQ_2 = Q_2$ and so

$$Q_2^*A = Q_2^*B^* = Q_2^*. \tag{5.34}$$

Rewriting this we get

$$Q_2^*(A - B^*) = Q_2^*(I_m - A) = Q_2^*(I_m - B^*). \tag{5.35}$$

If $m \leq n-r$, then Q_2^* has rank m which implies $I_m = A = B$, $X = I_{m+n-r}$ and

$$\mathcal{B} = \mathcal{B}'' = \{I_{m+n-r}\}' = M_{m+n-r}. \tag{5.36}$$

On the other hand if $m > n-r$, then there exist $m-n+r$ linearly independent solutions to the equation $Q_2^* v = 0$. These can be used to construct matrices $A, B \neq I_m$ that solve equation (5.35). Hence, $\mathcal{B} \neq M_{m+n-r}$. □

Theorem 5.2.6. *Let $r \geq 1$. The C^*-envelope of $H_B^\infty/\mathcal{I}$ is M_{m+n-r} if and only if $m \leq n-r$.*

Proof. This follows from Hamana's theorem and the fact that M_{m+n-r} is simple. □

As a corollary we obtain the following theorem from [15].

Corollary 5.2.7 ([15, Theorem 5.3])**.** *Let $z_1 = 0$ and $n \geq 3$. The C^*-envelope of $H_1^\infty/\mathcal{I}$ is M_{n+1}.*

Proof. Since $r = 1$ and $n \geq 3$ we see that $n - r = m = 2$. Hence, by the previous result $C_e^*(H_1^\infty/\mathcal{I}) = M_{m+n-r} = M_{n+1}$. □

As a corollary we also obtain Solazzo's result [38], which we stated as Theorem 5.2.2, for the algebra $H^\infty_{0,\frac{1}{2}}$.

To close our discussion we want to make a few statements about the relevance of the result on C^*-envelopes in distinguishing between scalar-valued and matrix-valued problems.

Note that the collection of one-dimensional subspaces of $H^2 \ominus BH^2$ can be identified with the complex-projective m-sphere PS^m. For a point $x \in PS^m$ let us denote by V_x the corresponding one-dimensional subspace of $H^2 \ominus BH^2$ and let the kernel for $V_x \oplus BH^2$ be denoted K^x. For a fixed pair of points $z, w \in \mathbb{D}$, the map $(z, w) \mapsto K^x(z, w)$ is continuous. Denote by $\mathcal{K}_x$ the span of the kernel functions at the points $z_1, \ldots, z_n$ for H_x^2. The interpolation theorem tells us that there is an isometric representation of $H_B^\infty/\mathcal{I}$ on $C(PS^m, M_{(n-r)+1})$ given by $\sigma(f + \mathcal{I})(x) = P_{\mathcal{K}_x} M_f P_{\mathcal{K}_x}$. If σ is a completely isometric representation, then $\mathcal{C} = C^*(\sigma(H_B^\infty/\mathcal{I}))$ is a candidate for $C_e^*(H_B^\infty/\mathcal{I})$. However, the C^*-algebra $\mathcal{C}$ is a subalgebra of $M_{(n-r)+1}(C(X))$ and as such its irreducible representations can be at most $(n - r + 1)$-dimensional. The fact that $m \geq 2$ tells us that $m + (n - r) > (n - r) + 1$ and this implies that $C_e^*(H_B^\infty/\mathcal{I})$ cannot be contained completely isometrically in $M_{(n-r)+1}(C(X))$. This contradiction proves that the matrix-valued analogue of the interpolation result in Theorem 4.1.1 is generally false.

One of the major results in [28] is the fact that $C_e^*(H^\infty(\mathbb{A})/\mathcal{I}) \cong M_n(C(\mathbb{T}))$ for $n \geq 3$. This is a result of a careful examination of the representation that arises from the Abrahamse-Ball theorem. For the space H_Γ^∞ one might naively guess that $C_e^*(H_\Gamma^\infty/\mathcal{I})$ is isomorphic to $M_n(C(\hat{\Gamma}))$. Given that the $p \times p$ matrix-valued interpolation theorem is parameterized by the orbit space $\mathrm{Hom}(\mathbb{F}_m, \mathcal{U}_p)/\mathcal{U}_p$, one might conjecture that $C_e^*(H_\Gamma^\infty/\mathcal{I}) \cong M_n(C^*(\Gamma))$. The results in Theorem 4.2.8 show that, even for an infinite, amenable group, the kernel on H_Γ^2 could be a complete Nevanlinna-Pick kernel. A consequence of this fact is that $C^*_e(H^\infty_\Gamma/\mathcal{I}) \cong M_n$ in this case. Hence, both conjectures above are generally false.

For the group Γ that arises from the covering of the annulus, Theorem 4.2.19 shows that there is a completely isometric representation of $H^\infty_\Gamma/\mathcal{I}$ on the span of the Szegö kernel functions $\left\{k_{\gamma^{(i)}(z_j)} : i \in \mathbb{Z}, j = 1, \ldots, n\right\}$. Therefore, $M_n(C(\mathbb{T}))$ is a quotient of the C^*-algebra generated by the image of this representation. It might be of value to know what this C^*-algebra is, but an explicit description of it seems beyond us at this time.

5.3 Topics to explore

It is fair to say that the theory of constrained problems still requires a lot of development. The study of these problems is of value. These algebras make connections both with function theory on multiply connected domains and with function theory in several complex variables.

The algebras H^∞_B provide us with an interesting class of naturally occurring finite co-dimension subalgebras of H^∞. In order to understand the finite-dimensional quotients as operator algebras we must have a more complete understanding of the matrix-valued interpolation problem. This would seem to be the next logical step. A good way to begin this process would be to explore the structure of the C^*-envelope for quotients of algebras of the form $H^\infty_{B_1} \cap H^\infty_{B_2}$ for two finite Blaschke products B_1 and B_2. At the other extreme would be a better understanding of the C^*-envelope of the quotient of H^∞_Γ.

The C^*-envelope approach has the drawback that it is not refined enough to distinguish any structure in the 2-point problem. In fact, when $n = 2$, the C^*-envelope is always either M_2 or $\mathbb{C} \oplus \mathbb{C}$! In these cases the correct approach would seem to be computing the Carathéodory metric for the algebra H^∞_B. This has been done for H^∞_1 by Knese [25] and was reformulated as a two-variable complex optimization problem in [15].

Structural results related to the maximal ideal space, corona problems and Bass stable rank of H^∞_B have been obtained by Mortini, Sasane and Wick [29].

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